On Parabolic Equations with Gradient Terms

Asma Elbirki

A thesis submitted for the degree of Doctor of Philosophy

in the

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I dedicate this thesis to

the memory of my father and my mother

For their endless love, support and encouragement.

Declaration

I hereby declare that this thesis has not been and will not be submitted in whole or in part to another university for the award of any other degree.

Signature:

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Abstract

This thesis is concerned with the study of the important effect of the gradient term in parabolic problems. More precisely, we study the global existence or nonexistence of solutions, and their asymptotic behaviour in finite or infinite time. Particularly when the power of the gradient term can increase to the power function of the solution. This thesis consists of five parts.

(i) Steady-State Solutions,

(ii) The Blow-up Behaviour of the Positive Solutions,

(iii) Parabolic Liouville-Type Theorems and the Universal Estimates,

(iv) The Global Existence of the Positive Solutions,

(v) Viscous Hamilton-Jacobi Equations (VHJ).

Under certain conditions on the exponents of both the function of the solution and the gradient term, the nonexistence of positive stationary solution of parabolic problems with gradient terms are proved in (i).

In (ii), we extend some known blow-up results of parabolic problems with perturbation terms, which is not too strong, to problems with stronger perturbation terms.

In (iii), the nonexistence of nonnegative, nontrivial bounded solutions for all negative and positive times on the whole space are showed for parabolic problems with

a strong perturbation term. Moreover, we study the connections between parabolic Liouville-type theorems and local and global properties of nonnegative classical solutions to parabolic problems with gradient terms. Namely, we use a general method for derivation of universal, pointwise a priori estimates of solutions from Liouville type theorems, which unifies the results of a priori bounds, decay estimates and initial and final blow up rates.

Global existence and stability, and unbounded global solutions are shown in (iv) when the perturbation term is stronger.

In (v) we show that the speed of divergence of gradient blow up (GBU) of solutions of Dirichlet problem for VHJ, especially the upper GBU rate estimate in n space dimensions is the same as in one space dimension.

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Contents

Chapter 1

Introduction

The latest developments of reaction diffusion systems, including gas dynamics, fluid mechanics, heat mass transfer, biology, ecology, engineering and many more, lead to wide studies in several aspects of nonlinear parabolic and elliptic partial differential equations. Because of the fast growth of reaction diffusion type of problems in various different fields it is desirable to have a unified mathematical treatment and practical methods for solving these problems.

Concerning the parabolic problems, it is known that many of these problems have global solutions, particularly, stationary solutions. On the contrary, the solutions of these problems may cease to exist in finite time. This is what we call singularities. The most marked property that recognize nonlinear partial differential equations from the linear partial differential equations is, the possibility of the formation of singularities from perfectly smooth initial data. In other words, for classes of data the theory of existence and uniqueness and continuity can be proved in small time intervals (well-posedness). However, in linear problems, the singularities of the solutions occur by singular coefficients or singular data of the problem, so-called fixed singularities.

The occurrence of singularities in nonlinear problems may depend on the nonlinearity of the terms in the problem and their time and location, which are called moving singularities. The easiest way to describe the spontaneous singularities in nonlinear problems is, when the variable or variables go to infinity as time tends to a certain finite time. This form of singularity is called a **blow-up phenomenon**. One of the most important examples of blow-up phenomenon is the heat equation with a source, namely,

$$u_t - \Delta u = F(x, t, u, \nabla u), \tag{1.1}$$

where the variable u represents the temperature in a chemical reaction, the second order derivative Δu appears as the diffusion, and the positive F is viewed as the heat source.

When high temperatures speed up the chemical reaction, the chemical reaction will generate more heat. As a consequence of this, the temperature will be very high, unless the heat energy become dissipative through the processes of the diffusion.

For instance, in solid fuel ignition, there is a competition between the heat source and the diffusion, which may lead to unbounded temperature in finite time. Physically speaking, the major increase of the temperature raises ignition. Hence, basic questions can be asked like:

1. Will blow-up occur?
2. Where are the blow-up points if the solution has to blow up in finite time?
3. What are the blow-up rate and the asymptotic behaviour of the solution near the blow-up time?
4. Will the solution continue after a finite time blow-up occurs?

These questions will be discussed in the next section.

In this thesis, we consider problems with nonlinearities depending on u and its space derivatives

$$\left.\begin{aligned} u_t - \Delta u = F(u, \nabla u), \qquad & x \in \Omega,\ t > 0 \\ u(x) = 0, \qquad & x \in \partial\Omega,\ t > 0, \\ u(x, 0) = u_0(x), \qquad & x \in \Omega, \end{aligned}\right\} \tag{1.2}$$

where $F = F(u, \xi) : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ is a C^1-function (except the problem (1.6) below with $1 < q < 2$).

In *Chapter* 2, we prove the nonexistence of positive radial ground states of the quasilinear elliptic equation

$$\Delta u + u^p - |\nabla u|^q = 0 \quad \text{in } \mathbb{R}^n,$$

when $p \leq 1 + 2/n$, $p > q > 2p/(p+1)$ and $n \geq 1$.

In *Chapter* 3, we consider the problem

$$\left.\begin{aligned} u_t &= \Delta u - h(|\nabla u|) + f(u) && \text{in } \Omega \times (0,T) \\ u &= 0 && \text{on } \partial\Omega \times (0,T) \\ u(x,0) &= u_0(x) && \text{in } \Omega, \end{aligned}\right\} \tag{1.3}$$

in the special case when $f(u) = u^p$, for $p > 1$, $p > q > 2p/(p+1)$,

$$f \in C^1([0,\infty)), \quad f(u) > 0 \quad u > 0,$$

$$h \in C^1([0,\infty)), \qquad h(v) > 0, \quad h^{'}(v) \geq 0 \quad \text{for} \quad v > 0, \quad \text{and}$$

$$\tilde{h}(v) := vh^{'}(v) - h(v) \leq kv^q \quad \text{for} \quad v > 0 \quad \text{and some} \quad 0 \leq k < \infty, q > 1. \tag{1.4}$$

We study the blow-up set for the problem (1.3) when Ω is a convex bounded domain, and we establish the blow-up rate estimate when Ω is a ball and Ω is a convex bounded domain.

In *Chapter* 4, we prove the Liouville-type theorems for the parabolic problem

$$u_t - \Delta u = u^p - \mu|\nabla u|^q, \quad x \in \mathbb{R}^n, \ t \in \mathbb{R},$$

when $p > q > 2p/(p+1)$ in radial case when $p < 1 + 2/n$, $n \geq 1$, and in general (nonradial) case when $p < n(n+2)/(n-1)^2$, $n \geq 2$. Moreover, in *Chapter* 4, in case $q = \frac{2p}{p+1}$ we derive the universal a priori bound for global solutions and usual blow-up rate estimates for the problem

$$\left.\begin{aligned} u_t - \Delta u &= u^p + g(u, \nabla u), && x \in \Omega, \quad 0 < t < T \\ u &= 0, && x \in \partial\Omega, \quad 0 < t < T, \end{aligned}\right\}$$

$$p < p_B, \quad \text{or} \quad p < p_S, \quad \Omega = \mathbb{R}^n \text{ or } B_R, \quad u = u(|x|, t), \quad g = g(u, |\xi|), \tag{1.5}$$

and the function $g : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}$ satisfies the growth assumption

$$|g(u, \xi)| \leq C_0(1 + |u|^{p_1} + |\xi|^q), \quad \text{for some} \quad 1 \leq p_1 \leq p.$$

Furthermore, in *Chapter* 4 we establish the blow-up estimates and a priori bounds of global solutions for the problem

$$\left.\begin{array}{ll} u_t - \Delta u = u^p - \mu|\nabla u|^q, & x \in \Omega, 0 < t < T, \\ u(x, t) = 0, & x \in \partial\Omega, 0 < t < T, \end{array}\right\}$$

in the case $q > 2p/(p + 1)$ and where Ω is a convex bounded domain.
In *Chapter* 5, we consider two problems, are given by

$$\left.\begin{array}{ll} u_t - \Delta u = u^p - a \cdot \nabla(u^q), & x \in \mathbb{R}^n, t > 0 \\ u(x, 0) = u_0(x), & x \in \mathbb{R}^n. \end{array}\right\}, \tag{1.6}$$

with $p > 1$, $q \geq 1$, a is a non zero constant vector in $\mathbb{R}^n$ and

$$\left.\begin{array}{ll} u_t - \Delta u = u^p - \mu|\nabla u|^q & , x \in \Omega, t > 0 \\ u(x, t) = 0 & , x \in \partial\Omega, t > 0 \\ u(x, 0) = u_0(x) & , x \in \Omega. \end{array}\right\}, \tag{1.7}$$

with $p, q > 1$ and $\mu > 0$.
Under certain conditions, we prove that the blow-up occurs just in infinite time for both problems (1.6) and (1.7).
In *Chapter* 6, we study the upper gradient blow-up rate estimate for the problem

$$\left.\begin{array}{ll} u_t - \Delta u = |\nabla u|^p + \lambda, & x \in \Omega, \quad t > 0, \\ u = 0, & x \in \partial\Omega, \quad t > 0, \\ u(x, 0) = u_0, & x \in \Omega, \end{array}\right\}$$

where Ω is a bounded convex domain, $p > 2$, $\lambda > 0$ and $u_0 \geq 0$.

1.1 Background

In the theory of ordinary differential equation (ODE), the blow-up are also analysed by *Osgood* in 1898 (see [**40**]), he showed that, the positive solution of the equation

$$u_t = f(u), \tag{1.8}$$

will blow-up in finite time for any positive initial data, provided f is defined for all sufficiently large $u \in \mathbb{R}$ and positive, and satisfies

$$\int_M^\infty \frac{\mathrm{d}u}{f(u)} < \infty, \tag{1.9}$$

for some $M > 0$. This is Osgood's criterion, and it is the necessary and sufficient condition for occurrence of blow-up in finite time. The simplest example is the initial value problem

$$u_t = u^2, \quad t > 0; \qquad u(0) = a, \qquad a > 0.$$

We see that the unique solution takes the formula $u(t) = \frac{1}{(T-t)}$, and it exists in the time interval $0 < t < T = 1/a$. It is obvious that the solution is a smooth function for $t < T$ and $u(t) \to \infty$ as $t \to T^-$. Thus, we say that the solution $u(t)$ has blow-up in finite time T.

Blow-up in reaction diffusion equations, for example the equation (1.1), which have a spatial structure, their solutions $u(x,t)$ depend on the time in the interval $0 \le t < T$ and space variable $x \in \Omega \subseteq \mathbb{R}^n$. Obviously, the spatial structure of the solutions is the new element of consideration, which distinguish the evolution equation (1.1) from the ordinary differential equation (1.8).

The idea of blow-up is built on the following points

1. The well-posedness of the parabolic problem in a certain space and for small times. The existence and uniqueness theory for the Cauchy problem or for the initial-boundary value problems in a certain class of bounded and nonnegative data can be found by assuming nice regularity conditions on ∇u and F in (1.1), in order to construct bounded solutions for some time $0 < t < T$.

2. The regularity and continuation theory are involved in this structure, which means that the bounded solutions have the necessary smoothness in order to continue locally in time. Schauder estimates are used for proving this theory for classical solutions of parabolic problems. This theory is also based on corresponding estimates which exist for weak solutions in a Sobolev space.

In a more general structure, the study of blow-up can be considered as a particular type of singularity that develops for a specific evolution process. For example, the evolution process, which is described by the equation and the initial condition

$$u_t = \mathcal{A}(u) \quad \text{for } t > 0, \qquad u(0) = u_0.$$

Consider the solution $u = u(t)$ as a curve living in a certain functional space X. Commonly, it can be proved that the problem is well-posed for some time $0 < t < T$ (i.e., the solution with the initial data u_0 is well-defined and lives in the space X), and the issue here is that the solution may leave the space when t approaches T^-. Now, from the previous illustration, we can define L^∞ blow-up as follows.

Definition 1.1.1. *We say that a solution u of (1.1) blows up at $t = T$ if the solution becomes infinite at some points or many points when t tends to the certain finite time T, which is called the blow-up time. In other words, there is a finite time T such that the solution is well defined for all $0 < t < T$ and*

$$\sup_{x\in\Omega} |u(x,t)| \to \infty \qquad \textit{as} \qquad t \to T^-.$$

In the next two definitions it can be seen that there are another situations where the reaction diffusion equations do not have global solutions.

Definition 1.1.2. *If the reaction term of the reaction diffusion equation (1.1) is singular for finite u in finite time T, i.e., $f \to \infty$ as $u \to K$ for some $K > 0$, then the reaction term blows up, and the smooth solution ceases to exist. This phenomenon is called quenching or extinction.*
In fact, it can be shown that under suitable assumptions, quenching implies blow-up of u_t, namely $\lim_{t\to T^-} \|u_t\|_\infty = \infty$.

A typical example of quenching phenomena is (1.1) with $F(u) = -u^{-p}$, $p > 0$. It is possible that there exists a finite time T such that $u \to 0$ as $t \to T$ (see [**50**]).

Definition 1.1.3. *If the solution of the reaction diffusion equation (1.1) can cease to exist in finite time T only when*

$$\lim_{t \to T^-} \|\nabla u(t)\|_\infty = \infty,$$

this is called gradient blow-up (GBU for short).

For instance, if the reaction term in (1.1) depends only on the spatial gradient, GBU can occur, even if the solution stays bounded (see [**47**]).

Remark 1.1.4. *The blow-up for elliptic and other stationary solutions of the corresponding parabolic equations are studied by many authors, they found that the singularity can also occur at a certain point (or points).*

Next, we analyse some existing questions from the literature of blow-up for reaction diffusion equations.

1.1.1 Will the blow-up occur in finite time?

Where the existence and uniqueness can be commonly proved in different functional spaces, and the blow-up occurs in that space when the solution cannot continue up to or past a given time, the blow-up might occur in a one function space and not in another one. This shows the rich structure of blow-up problems. The above question can be answered in two directions:

i. The form of the problem (the equation's coefficients and nonlinearities or more widely its structural conditions), and also the form of the initial data. For instance in [**54**] it has been shown that the Dirichlet problem for the parabolic problem

$$\left.\begin{aligned} u_t &= \Delta u - \mu|\nabla u|^q + u^p && \text{in } \Omega \times (0,T) \\ u(x,t) &= 0 && \text{on } \partial\Omega \times (0,T) \\ u(x,0) &= u_0(x) && \text{in } \Omega, \end{aligned}\right\} \tag{1.10}$$

in bounded domain has a finite-time blow-up for large initial data when $p > q$. It was also proved that, the condition $p > q$ is optimal, which guarantee that the solution blows up in finite time in bounded domains (see [**16**],[**46**]). However, for general unbounded domains the issue depends on the geometry of the domain, through the notion of the inradius $\rho(\Omega)$ (see Chapter 6). It was shown that the solution of the problem (1.10) is globally bounded for all nonnegative data if $\rho(\Omega) < \infty$, and if $\rho(\Omega) = \infty$ the solution will either blow up in finite time or will be an unbounded global solution (blow-up in infinite time). The blow-up can be also generated by the nonlinearity of the boundary conditions, for example, in [**27**] the Neumann problem

$$\left.\begin{aligned} u_t &= \Delta u, && \text{in } \ B_R \times (0,T) \\ \frac{\partial u}{\partial \nu} &= f(u), && \text{on } \ \partial B_R \times (0,T) \\ u(x,0) &= u_0(x), && \text{in } \ B_R, \end{aligned}\right\} \tag{1.11}$$

has been considered, where $B_R \subset \mathbb{R}^n$ is a ball of radius R, ν is the unit exterior normal vector, and $f \in C^1$ is nondecreasing and satisfies (1.9). It was shown that, for all $u_0 \geq 0$, the solution u blows up in a finite time.

ii. If the problems do exhibit blow-up in finite time, the question here is, which solutions do blow up in finite time? The answer is either all solutions of the problem blow up in finite time, which is called *Fujita problem* or for some identified solutions. For instance, the Fujita results have been given in [**37**] for the problem

$$\left.\begin{aligned} u_t &= \Delta u - \mu|\nabla u|^q + u^p && x \in \mathbb{R}^n, \ t > 0 \\ u(x,0) &= u_0(x) && x \in \mathbb{R}^n, \end{aligned}\right\} \tag{1.12}$$

which are, if $p > 1$, $q = 2p/(p+1)$ and $p < 1 + 2/n$, then the solution blows up in finite time for all $u_0 \geq 0$. However, the global existence of the solution for small initial data is obtained when $p > 1 + 2/n$ (see [**47**]).

In the case when the global existence of the solution can be found in a functional space for all $0 < t < \infty$, but the solution becomes unbounded when $t \to \infty$, we say

that the solution blows up in infinite time, and the solution in this case is called **unbounded global solution**. For example, it is known in [**55**] that for the problem (1.12) with $q \geq p > 1$ and $\mu > 0$, there exists $u_0 \geq 0$ with compact support, such that the solution u blows up in infinite time and u is unbounded (even $\lim_{t\to\infty} u(x,t) = \infty$ for all $x \in \mathbb{R}^n$). Moreover, there is another case of blow-up, in which the solution becomes unbounded ($u(x,t) = \infty$) for any arbitrarily small $t > 0$, this means that the complete singularity happens at $t = 0$ and the nonexistence of nontrivial solution is locally in time, this case is called **instantaneous blow-up**. For instance in the latter case, in [**42**] the exponential reaction equation $u_t = \Delta u + \lambda e^u$ was considered in a bounded domain $\Omega \subset \mathbb{R}^n$ with initial data u_0 and $u = 0$ on the smooth boundary $\partial\Omega$. It was proved that the solution of this equation has instantaneous blow-up, if $u_0 \geq S(x) = -2\ln|x|$, $u_0 \not\equiv S$ and $n \geq 10$.
Therefore there are alternative options which can occur for the solution of reaction diffusion problems, and they can be classified as follows:

(i) **Bounded global solutions**, which means that the solution remains uniformly bounded in time, i.e., no blow-up occurs.

(ii) **Unbounded global solutions**, i.e., the solution blows-up at infinity.

(iii) **Finite-time blow-up solutions**, the solution becomes unbounded at a finite time T.

(iv) **Instantaneous blow-up solutions**, which the solution blows-up at $T = 0$.

Remark 1.1.5. *We mean by the global solution the case (i) or (ii). Furthermore, the case (iii) is called the standard blow-up case.*

Next it comes to the second question, which is:

1.1.2 Where are the blow-up points if the blow-up occurs at a time $T < \infty$ or at $T = \infty$?

If the solution u blows-up in a finite time T or in infinite time "$T = \infty$", the blow-up set $B(u_0)$ which is closed and subset of $\overline{\Omega} \subseteq \mathbb{R}^n$ is defined as follows:

Definition 1.1.6. *We say that $x \in \Omega$ is a blow-up point, if there exists a sequence $\{x_n\}$ and $t_n \to T$ such that $x_n \to x$ and $|u(x_n, t_n)| \to \infty$. In other words,*

$$B(u_0) = \{x \in \overline{\Omega} \cup \{\infty\} : \; \exists \{x_n, t_n\} \subset \Omega \times (0, T), t_n \to T^-, x_n \to x, \\ |u(x_n, t_n)| \to \infty\}.$$

Remark 1.1.7. *In case of gradient blow-up phenomena, where the gradient of solution ∇u blows up in finite time T, the blow-up set is*

$$B(u_0) = \{x \in \overline{\Omega} : \; \exists \{x_n, t_n\} \subset \Omega \times (0, T), t_n \to T^-, x_n \to x, \; |\nabla u(x_n, t_n)| \to \infty\}.$$

It was proved in [**21**] that for the problem (1.10) in the radially symmetric case, if $u_0 \geq 0$ be radial nonincreasing, $1 < q < p$, $\mu > 0$ and $\Omega = B_R$, then 0 is the only blow-up point, which is called a single point blow-up. Also, it has been shown in [**13**] that if Ω is convex and bounded, $\mu > 0$ and $q < 2p/(p+1)$, then the blow-up set of any solution of (1.10) is a compact subset of Ω, which is called regional blow-up. On the other hand, when $\mu < 0$ in the problem (1.10) the situation is different, the blow-up set consists of a single point when $q = 2$ and $p > 1$, regional blow-up when $p = 2$, and global blow-up when $1 < p < 2$ (see [**33**], [**31**], [**23**]).
Therefore, the blow-up set $B(u_0)$ can be classified as one of the following ways:

i. **Single-point blow-up**, where $B(u_0)$ contains a single point or of a finite number of points.

ii. **Regional blow-up**, in which the measure of $B(u_0)$ is finite and positive.

iii. **Global blow-up**, where $B(u_0) = \overline{\Omega}$.

Furthermore, the blow-up set of unbounded global solutions of the parabolic problem (1.10) when $q \geq p > 1$, have been studied in [**55**]. It was shown that, if $\Omega = \mathbb{R}^n$, then there exists u_0 compactly supported such that the solutions blow up everywhere in infinite time, i.e., $B(u_0) = \mathbb{R}^n$. Moreover, it was proved that if $\Omega \subset \mathbb{R}^n$ (unbounded), $q \geq p > 1$, $\mu > 0$ and u_0 is chosen such that either $T = \infty$ or $T < \infty$, then $B(u_0) = \{\infty\}$ if $\Omega \neq \mathbb{R}^n$ and $B(u_0) = \mathbb{R}^n \cup \{\infty\}$ or $B(u_0) = \{\infty\}$ if $\Omega = \mathbb{R}^n$ (see [**55**]).

The next question is:

1.1.3 What are the blow-up rate and the asymptotic behaviour of the solution near the blow-up time?

When the solution of parabolic problem blows-up in a finite time, one might ask about the space-time behaviour of the blow-up solution when t approaches T and near or at blow-up points. There are several aspects to study the space-time behaviour of the blow-up solutions.

(i) **Blow-up rate estimate**:

It is the rate at which the solution u diverges as t approaches the blow-up time and x approaches a blow-up point.

Where the nonlinearity in parabolic problems causes the blow-up in a finite time in certain cases. Furthermore, the solution to the ODE

$$u_t = u^p, \quad t > 0,\ p > 1, \quad u(0) = u_0 > 0, \tag{1.13}$$

is given by

$$u(t) = k(T-t)^{-1/(p-1)}, \ \ 0 < t < T, \quad \text{with } k = (p-1)^{-1/(p-1)}, \tag{1.14}$$

and $T = (p-1)^{-1}u_0^{1-p}$. The blow-up rate for ODE is $k(T-t)^{-1/(p-1)}$. If we consider the problem (1.10) and we assume that $T < \infty$, a natural question is whether the blow-up for the model problem (1.10) will be the same as that for the corresponding ODE. In other words, are the diffusion and gradient term weak enough to have no impact on the blow-up rate estimate? Namely, the rate takes the form

$$C_1(T-t)^{-1/(p-1)} \leq \|u(t)\|_{L^\infty} \leq C_2(T-t)^{-1/(p-1)}, \tag{1.15}$$

where $C_1, C_2 > 0$. The answer depends on the structure of the problem, where for some problems, the blow-up rate for (1.10) can be different from the ODE, however, the rate estimates are of the same order for large class of the problems. When the blow-up rate estimate for the parabolic problems remains the same as for the corresponding ODE, the estimate is referred to be of **type I**, otherwise, blow-up is referred to as **type II**. For instance, In [**13, 12, 18, 58**], under some assumptions on Ω and p, the rate estimates of the blow-up solution of the problem (1.10) were proved in the case $q < 2p/(p+1)$, and the blow-up is of the type I, and it takes the form

$$C_1(T-t)^{-1/(p-1)} \leq \|u(t)\|_\infty \leq C_2(T-t)^{-1/(p-1)}, \qquad \text{as} \;\; t \to T, \tag{1.16}$$

where $C_1, C_2 > 0$. Moreover, we proved in this thesis in Chapter 3 that for stronger absorbing gradient term ($q > 2p/(p+1)$) in the problem (1.10), the blow-up rate becomes faster, or type II (see Theorem 3.3.5), and it takes the form

$$u(x,t) \leq C(T-t)^{\frac{-q}{2(p-q)}}, \quad t \in (0,T).$$

Moreover, it can be shown from the solution which constructed in the proof of Proposition 3.2 in [**55**] that, the solution u of the problem (1.10) which blows-up in infinite time, where Ω contains a cone, $\mu \geq 0$, $q \geq p > 1$ and $\phi \geq 0$ with compact support, it satisfies the estimate

$$C_1 \leq \|u(t)\|_\infty \leq C_2 e^{C_3 t}, \quad t \to \infty, \quad C_1,\ C_2,\ C_3 > 0.$$

(ii) **Final blow-up profile**:

It is asymptotic behaviour of the solution near the blow-up time, which is described as a limit of $u(x,t)$ when $t \to T^-$ at the non-blowing points.

In [**13**] an interesting result is proved concerning the blow-up profile for the solution of the problem (1.10) in the radial case. It was shown that, if the solution u blows-up in finite time, $1 < q < p$, $\mu > 0$, $\Omega = B_R$, and $u_0 \geq 0$ be radial nonincreasing, then 0 is the only blow-up point and the blow-up profile of the solution takes the form

$$u(r,t) \leq Cr^{-\alpha}, \qquad (r,t) \in (0,R] \times [0,T), \tag{1.17}$$

for any $\alpha > 2/(p-1)$ if $1 < q < 2p/(p+1)$, and for any $\alpha > q/(p-q)$ if $2p/(p+1) \leq q < p$. It was observed that the blow-up profile of the solution is similar to the problem (1.10) without gradient term ($\mu = 0$) when $1 < q < 2p/(p+1)$, while for $2p/(p+1) \leq q < p$, the gradient term has an important effect on the profile, which becomes more singular due to $q/(p-q) > 2/(p-1)$. Moreover, the space profile of GBU of the solution of the viscous Hamilton-Jacobi equation was studied in one space dimension by Quittner and Souplet [**47**]

$$\left.\begin{aligned} u_t - \Delta u = |\nabla u|^p, &\qquad x \in \Omega, \quad t > 0, \\ u = 0, &\qquad x \in \partial\Omega, \quad t > 0, \\ u(x,0) = u_0, &\qquad x \in \Omega, \end{aligned}\right\} \tag{1.18}$$

for $p > 2$ and $\Omega = (0,1)$, $u_0 \geq 0$ and $t_0 \in (0,T)$. They found that, the bound of GBU of u_x away from $x = 0$ and $x = 1$, is given by,

$$u_x(x,t) \leq (p-1)^{-1/(p-1)} x^{-1/(p-1)} + C_1 x, \qquad x \in (0,1]$$

and

$$u_x(x,t) \geq -(p-1)^{-1/(p-1)}(1-x)^{-1/(p-1)} - C_1(1-x), \qquad x \in [0,1),$$

for some $C_1 > 0$ and all $t \in (t_0, T)$. These profiles guarantee that $x = 0$ and $x = 1$ are the only possible GBU points.

Now we come to the final question which is referred to the continuation of the solution after a finite blow-up time as follows:

1.1.4 Will the solution continue after a finite time blow-up occur?

This question considers whether or not the blow-up solution can continue after a finite blow-up time in some weak sense. There are three possible cases that might happen after the occurrence of the blow-up in finite time:

i. **Complete Blow-up**, in which the solution cannot be continued after occurrence of the blow up, and if the solution is continued, it will be infinite everywhere.

ii. **Incomplete Blow-up**, in which the solution can be continued in subsets of the space-time after a finite blow-up T, and the solution is infinite in the complement of these subsets.

iii. **Transient Blow-up**, in which the solution becomes bounded again after the blow-up occurs at a finite time T.

For instance on the complete blow-up phenomenon, it has been shown in [**5**] that if $\mu = 0$ in the problem (1.10), $1 < p < p_S := \frac{n+2}{n-2}$ if $n \geq 3$ or $1 < p < \infty$ if $n = 1, 2$. Then, the continuation of the solution after T is not possible and the solution becomes infinite after a finite T. Moreover, it is known that under restrictive assumption, the solution of the problem (1.10) blows up incompletely when $p > p_S := \frac{n+2}{n-2}$, $\mu = 0$, Ω is bounded and $p > 1$ (see [**38, 24**]). Furthermore, in [**24**] a transient blow-up phenomenon was analysed for the slow diffusion equation $u_t = \Delta u^m + u^p$ for $x \in \mathbb{R}^n$, $t > 0$ and $p > m(n+2)/(n-2)$ for $n \geq 3$, $m > (n-2)/n$. It was proved that this problem has a radial solution which blows up at a single point peak at $t = T$, and then the solution becomes bounded immediately for the rest of the time $t > T$.

1.2 Outline of the Thesis

This thesis analyses some problems concerning the qualitative study of solutions of elliptic and parabolic equations with gradient terms, and with homogeneous Dirichlet boundary conditions. The main purpose of this thesis is to investigate the impact of the gradient terms on the global existence and nonexistence of the solutions, and on their asymptotic behaviour in finite and infinite time. A number of qualitatively new phenomena can appear by the presence of the gradient terms, in comparison with the their absence, such as behaviour of blowing-up solutions, global existence and stability, unbounded global solutions and critical exponents. The next four chapters are organised as follows:

Chapter 2 is devoted to study the steady-state solution of quasilinear elliptic equation with a dissipative gradient term, $\Delta u + u^p - |\nabla u|^q = 0$, defined in the whole space $\mathbb{R}^n$, when $p \leq p_F := 1 + \frac{2}{n}$ and $\frac{2p}{p+1} < q < p$. We prove the nonexistence of the radial ground state of this equation, which will be a useful result in the investigation of the corresponding parabolic problem. For this equation, we extend the known nonexistence results by Serrin and Zou when $\frac{2p}{p+1} < q < p$ and $p \leq \frac{n}{n-2}$, showing that the nonexistence can be also shown by using the asymptotic behavior of the solutions at infinity (see [**51**]).

Chapter 3 extends the results of Chlebik, Fila and Quittner [**13**] for the semilinear parabolic problem with a dissipative gradient term, under the condition $q < 2p/(p+1)$ to the stronger condition $q > 2p/(p+1)$. The blow-up sets and the upper blow-up rate estimates are derived for the problem in the radial symmetric case where the domain is a ball and in the convex domain case when $q > 2p/(p+1)$. It is proved that the set of the blow-up is a compact subset of the domain in the convex domain case. Moreover, we show that the stronger absorbing gradient term has an important effect on the upper blow-up rate estimates in both cases, which make them more singular than those known in [**13**] for $q < 2p/(p+1)$. Applying the maximum principle to suitable auxiliary function is used to determine the blow-up set in the convex domain case, and to derive the rate estimates in both cases.

Chapter 4 is devoted to study the parabolic Liouville-type theorems and the universal estimates. Two parabolic problems are considered in this chapter. Firstly, we study the parabolic equation with a dissipative gradient term $u_t - \Delta u = u^p - \mu|\nabla u|^q$, defined in the whole space or in a convex bounded domain. Secondly, we study the parabolic problem with first-order derivatives, namely, $u_t - \Delta u = u^p + g(u, \nabla u)$, defined in the whole space or in a ball. The Liouville-type theorems for first equation are proved in radial case for $p < p_F$ when $\frac{2p}{p+1} < q < p$, and in general (nonradial) case for $p < p_B := \frac{n(n+2)}{(n-1)^2}$, $n > 1$, when $\frac{2p}{p+1} < q < p$. The intersection-comparison is used to prove Liouville theorem in radial case, while integral estimates are used to prove the theorem in general case. Moreover, the universal a priori bounds for global solutions and usual blow-up rate estimates are derived for two cases, when $q = 2p/(p+1)$ for the second problem, and when $q > 2p/(p+1)$ for the first problem in a convex bounded domain. We use a doubling lemma and a rescaling argument to derive the estimate for the solution of the second problem when $q = 2p/(p+1)$. On the other hand, the method which is used to derive the estimate of the solution for the first problem when $q > 2p/(p+1)$ is based on energy, measure arguments, rescaling and elliptic Liouville-type theorems. We show that the estimate for $q = 2p/(p+1)$ takes the same form as these known for the problem when $q < 2p/(p+1)$ (see [**47**]). However, we show that the universal bounds when $q = 2p/(p+1)$ do not remain valid for the problem if the perturbation term is stronger, i.e., when $q > 2p/(p+1)$.
Chapter 5 considers the existence of the positive solutions for the parabolic equations with a gradient term. In this chapter, two special cases are studied. First case is, the semilinear parabolic equation with a convective gradient term $u_t - \Delta u = u^p - a \cdot \nabla(u^q)$ for the Cauchy problem. The semilinear parabolic problem with a dissipative gradient term in the domain of infinite inradius is studied in the second case. For the first problem, by building a self-similar supersolution for the problem with a convective gradient term and using a comparison principle, we show that the solution can exist globally when $q > p > 1$. For the problem with a dissipative gradient term and under the condition $p < p_F$, we use a technique

which depending on the rescaling arguments, to show that the solution is bounded for an initial data small enough. Furthermore, for positive initial data, global unbounded solutions are proved for the problem with a dissipative gradient term when $q \geq p > 1$.

Chapter 6 considers Viscous Hamilton-Jacobi Equations (VHJ) for $p > 2$. In this chapter, we consider a VHJ parabolic equation with nonlinear term depending on the gradient of the solution u, defined in a convex bounded domain $\Omega \subset \mathbb{R}^n$ and with a Dirichlet boundary condition. Namely, $u_t - \Delta u = |\nabla u|^p$, where $p > 2$. The blow-up phenomena here are different, where the gradient of the solution of this problem becomes unbounded in a finite time, while the solution itself remains uniformly bounded. The upper estimate of the gradient blow-up profile is considered, showing that the gradient blow-up may just occur on the boundary of the domain. We also use a suitable auxiliary function with application of the maximum principle to prove that the analogue of the upper GBU rate estimate in one dimension is still true in higher dimensions when $p > 2$.

In **Chapter 7**, the main results for every chapter are summarised in this chapter.

We collect a number of frequently used notations and results in four appendices:

Appendix A includes the geometric domain notation, definitions of radial functions, and revision for some standard function spaces.

In **Appendix B** we introduce some basic inequalities which are employed throughout this thesis.

In **Appendix C** we recall some fundamental estimates for elliptic and parabolic equations. Moreover, we review some definitions of classical solutions, maximal solutions, weak solutions, supersolutions and subsolutions.

In **Appendix D** we state some maximum and comparison principles, which we frequently used in this study.

1.3 Preliminaries

Through this thesis we denote by

$$X := \{u \in BC^1(\overline{\Omega}) : u = 0 \ \text{ on } \ \partial\Omega\},$$

provided with the norm

$$\|w\|_X := \|w\|_\infty + \|\nabla w\|_\infty ,$$

and $X_+ := \{w \in X : w \geq 0\}$. The general problem (1.2) with nonlinearities depending on u and its space derivative is locally well-posed in X, where $F = F(u, \xi) : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ is a C^1-function (except the problem (1.6) with $1 < q < 2$) (see [**47**]). Moreover, the problem (1.6) with $\Omega = \mathbb{R}^n$, which is also well-posed in X for all $q \geq 1$, with Ω bounded or $\Omega = \mathbb{R}^n$ (see [**47**]). Specifically,

$$\text{if } \ T_{max} = T_{max}(u_0) < \infty, \ \text{ then } \quad \lim_{t \to T_{max}} \|u(t)\|_X = \infty.$$

Furthermore, in [**47**] it has been shown that, the solution enjoys the regularity property

$$u \in BC^{2,1}(\overline{\Omega} \times [t_1, t_2]), \qquad 0 < t_1 < t_2 < T_{max}(u_0). \tag{1.19}$$

In particular, in the case of problem (1.7) is defined in a ball and u_0 is radial, or in $\mathbb{R}^n$ and u_0 is radial nonincreasing, then the solution u enjoy the same property (see [**47**]). Moreover, in this thesis, we will refer to the comparison principles, cf. Proposition D.1.2, Proposition D.1.7 and Remark D.1.3 without explicit reference. In addition, throughout this thesis we will use various critical exponents

$$p_S := \begin{cases} \frac{n+2}{n-2}, & \text{if } n \geq 3. \\ \infty, & \text{if } n = 1, 2. \end{cases}$$

$$p_B := \begin{cases} \frac{n(n+2)}{(n-1)^2}, & \text{if } n \geq 2. \\ \infty, & \text{if } n = 1. \end{cases}$$

$$p_F := 1 + \frac{2}{n}, \qquad n \geq 1.$$

Chapter 2

The Steady-State Solution of the Quasilinear Elliptic Equation

When the solution of (1.10) is independent of t, it is called steady-state solution, or stationary solution. This solution is the possible limit as $t \to \infty$ of the corresponding time-dependent solutions if the time-dependent solution is global. The ground states of the elliptic analogue which is the corresponding to the parabolic equation (1.10), is a positive solution u in $\mathbb{R}^n$, which tends to zero as $|x| \to \infty$.

Our aim here is to prove that the nonexistence of radial ground states of $\Delta u + u^p - |\nabla u|^q = 0$ will hold for $p \leq p_F$. Moreover, we show that, the nonexistence in this problem depends on the specific values of the exponents of the function of the solution and its gradient. In section 2.2 we present some preliminary results, which we need to determine the asymptotic estimates in section 2.3. Section 2.4 is devoted to the nonexistence of positive radial ground states for $p \leq p_F$ when $q > 2p/(p+1)$.

2.1 Introduction

This chapter is concerned with the quasilinear elliptic equation

$$\Delta u + u^p - |\nabla u|^q = 0 \quad \text{in } \mathbb{R}^n, \tag{2.1}$$

where p and q are given positive exponents and $n \geq 1$.

The steady states of (1.10), i.e., the solutions of the elliptic problem

$$\left.\begin{aligned} \Delta u + u^p - \mu|\nabla u|^q = 0, &\qquad x \in \Omega, \\ u(x) = 0, &\qquad x \in \partial\Omega, \end{aligned}\right\} \tag{2.2}$$

have been considered by many authors, e.g., [**3, 10, 11, 15, 17, 41, 51, 61**]. In [**26**], the authors obtained an interesting result; they prove that the solution of (2.2) on $\mathbb{R}^n$ or on a ball must be radial.

The interest in this problem arises particularly from the dependence of existence and nonexistence on the specific values $p,\ q,\ \mu$.

We are going to summarize the results about the existence and nonexistence of the solutions to (2.2) in the case $\Omega = \mathbb{R}^n$ and Ω is a ball B_R in $\mathbb{R}^n$. By "existence", we mean the existence of at least one classical positive solution of (2.2) on Ω.

First, when $\Omega = \mathbb{R}^n$, the following holds

i. If $p > p_S$, the existence can be found for all $q > 1$ (see [**51**]);

ii. If $p = p_S$, the existence is obtained if and only if $q < p$ (see [**51**]);

iii. If $p < p_S$

   (a) the existence can be proved if $q < 2p/(p+1)$ or $q = 2p/(p+1)$ and μ is large enough (see [**11**]);

   (b) nonexistence holds if $p \leq n/(n-2)$, $n > 2$ and $q > 2p/(p+1)$ (see [**51**]);

   (c) nonexistence holds if $p < n/(n-2)$, $n > 2$ and $q = 2p/(p+1)$ with μ small enough (see [**9, 17, 51**]);

(d) the nonexistence is proved if $n \geq 3$, $n/(n-2) < p < p_S$ and $q > \bar{q}$ for some $\bar{q} \in (2p/(p+1), p)$ (see [**51**])

Next, in the case $\Omega = B_R$ in $\mathbb{R}^n$, it holds:

i. The existence is proved if $1 < q < 2p/(p+1)$ and $p < p_S$ [**11**];

ii. If $q = 2p/(p+1)$ then

   (a) the nonexistence is obtained if $p \geq p_S$ [**51**] or if $p < p_S$ and μ is large [**11**];

   (b) the nonexistence can be proved if $p \leq n/(n-2)$ and μ is small enough [**11, 17, 61**];

iii. The existence can be found for μ small [**11**] and the nonexistence is proved for μ large [**10**], if $2p/(p+1) < q < p$ and $p < p_S$;

iv. If $q \geq p > 1$, the existence is satisfied, if and only if $\mu \leq \mu_0$, for some $\mu_0 = \mu_0(p, n) > 0$ [**46, 61**].

Moreover, there are some results, which are known when Ω is an arbitrary bounded domain with smooth boundary:

i. If $p < p_S$, the existence can be proved if μ is small enough [**61**];

ii. If $q \geq p > 1$, the existence is satisfied if and only if $\mu \leq \mu_0$, for some $\mu_0 = \mu_0(p, n)$ [**46, 61**].

The aim of this chapter is to extend the known nonexistence results in [**51**], for the elliptic equation with a gradient term when $q > 2p/(p+1)$ and $p \leq n/(n-2)$, showing that the nonexistence of the positive stationary solutions can also be proved when $q > 2p/(p+1)$ and $p \leq 1 + 2/n$.

2.2 Preliminary Results

In this chapter we consider the radial ground states $u(r)$ of (2.1), where $r = |x|$ is the radius. Clearly, $u(r)$ can be a solution of the following initial value problem

$$u''(r) + \frac{n-1}{r}u'(r) + u^p(r) - |u'(r)|^q = 0, \tag{2.3}$$

$$u(0) = \xi, \quad u'(0) = 0,$$

with

$$u(r) > 0 \quad \text{for all} \quad r > 0, \tag{2.4}$$

for some $\xi > 0$.

Furthermore, we also consider the solutions of (2.3) which do not satisfy (2.4), and satisfy the conditions

$$u(r) > 0 \ \text{ for } \ 0 \leq r < R, \quad u(R) = 0. \tag{2.5}$$

Before proving the nonexistence of ground states of (2.1), we need to find the asymptotic behaviour of the solution of (2.3) at infinity, which is crucial for the argument of the proof in Section 2.4. The proof of these estimates depend on some preliminary lemmas which we will introduce in this section.

Now, concerning the proof of Theorem 2.2.3 below, we just need to recall Lemmas 2.2.1 and 2.2.2, which have already been proved in [**51**].

Lemma 2.2.1. [**51**] *Suppose that $R > 1$, $p > 1$, $2p/(p+1) \leq q < p$ holds, and we define the set*

$$A = \mathit{interior}\{1 < r < R \mid S(r) \geq 0\}, \tag{2.6}$$

where $S(r) = S_\gamma(r) = u^p - 2|u'|^\gamma, \quad 0 < r < R, \quad \gamma > 0$. Then

$$u \leq C_2 r^{-2/(p-1)}, \quad r \in A \tag{2.7}$$

with

$C_2 = 4^{2/(p-1)} \max\{\xi, C_1, [8n/(p-1)]^{1/(p-1)}\}, \quad C_1 = [2q(n-1)/p]^\sigma$ *and* $\sigma = q/(p-q)$.

Lemma 2.2.2. [**51**] *Suppose that* $R > 1$, $p > 1$, $2p/(p+1) \le q < p$ *holds, and we define the set*

$$B = \{1 < r < R \mid S(r) < 0\}, \tag{2.8}$$

where $S(r) = S_\gamma(r) = u^p - 2|u'|^\gamma, \quad 0 < r < R, \quad \gamma > 0$. *Then*

$$u \le C_3 r^{-2/(p-1)}, \quad r \in B, \tag{2.9}$$

where

$$C_3 = \max\{(4\sigma)^\sigma,\ 4^{1/(p-1)} C_2\}.$$

We have the following theorem

Theorem 2.2.3. *Suppose that* $p > p_F$, $\quad n \ge 1$ *and* $2p/(p+1) \le q < p$. *Then*

$$u \le C_3 r^{\frac{-2}{p-1}}, \tag{2.10}$$

$$|u'|^{q-1} \le C_4^* r^{-1-\delta}, \quad 0 < r < R, \tag{2.11}$$

$$\text{where} \qquad \delta = \frac{(p+1)q - 2p}{p-1} \ge 0, \quad C_4^* = \Big[\frac{(p-1)C_3^p}{n(p-1)-2}\Big]^{q-1}.$$

Proof. The estimate (2.10) follows directly from (2.7) and (2.9) since

$$\overline{A \cup B} = [1, R] \quad \textit{and} \quad u \le \xi r^{\frac{-2}{p-1}} \quad \text{when} \quad 0 < r < 1.$$

To prove (2.11), multiply (2.3) by r^{n+1} and integrate from zero to r to get

$$\begin{aligned}
r^{n+1}|u'| &= \int_0^r s^{n+1} u^p \mathrm{d}s - 2\int_0^r s^n u' \mathrm{d}s - \int_0^r s^{n+1}|u'|^q \mathrm{d}s \\
&\le \int_0^r s^{n+1} u^p \mathrm{d}s + 2n \int_0^r s^{n-1} u \mathrm{d}s \\
&\le C_3^p \Big[\int_0^r s^{n+1-\frac{2p}{p-1}} \mathrm{d}s + \int_0^r s^{n-1} u \mathrm{d}s\Big] \\
&= \frac{(p-1)C_3^p}{n(p-1)-2}\, r^{n-\frac{2}{p-1}}
\end{aligned}$$

since $p > 1 + \frac{2}{n}$, it follows

$$|u'|^{q-1} \le \Big[\frac{(p-1)C_3^p}{n(p-1)-2}\Big]^{(q-1)} r^{(q-1)(-1-\frac{2}{p-1})} = C_4^* r^{-1-\delta}, \qquad 0 < r < R,$$

where $\delta = \frac{(p+1)q-2p}{p-1} \ge 0, \quad C_4^* = [\frac{(p-1)C_3^p}{n(p-1)-2}]^{q-1}$. $\square$

2.3 Asymptotic Behaviour of Radial Solutions

In this section, we study the asymptotic behaviour at infinity of the solutions of (2.3) which satisfy (2.4). This asymptotic behaviour will be essential to prove the nonexistence of ground states of the problem (2.1).

We recall now the following theorem from [**51**], which gives the upper asymptotic estimate for ground states of the problem (2.3). One of these estimates is important to prove Theorem 2.3.2 and the nonexistence in the next section.

Theorem 2.3.1. [**51**] *Suppose that $u(r)$ is a solution of (2.3) which satisfies (2.4) and that $p \geq 1$, $2p/(p+1) < q < p$ holds. Then*

1. *If $p < 1$, there is no solution.* *(2.12)*
2. *If $p = 1$, then $u = \mathcal{O}\Big(e^{-r}\Big)$, $u' = \mathcal{O}\Big(e^{-r}\Big)$ as $r \to \infty$.* *(2.13)*
3. *If $p > 1$, then $u = \mathcal{O}\Big(r^{-\frac{2}{p-1}}\Big)$, $u' = \mathcal{O}\Big(r^{-\frac{p+1}{p-1}}\Big)$ as $r \to \infty$.* *(2.14)*

The next theorem considers the upper asymptotic estimate of u' for the problem (2.3).

Theorem 2.3.2. *Suppose that $u(r)$ is a solution of (2.3) satisfying (2.4), $1 \leq p \leq p_F$ or $p > p_F$. Then*

$$u' = \mathcal{O}(r^{-\frac{p+1}{p-1}}) \tag{2.15}$$

and

$$\int_0^\infty |u'|^{q-1} \mathrm{d}s < \infty. \tag{2.16}$$

Proof. i. Assume $p > p_F$. Equations (2.15) and (2.16) follow directly from the argument of Theorem 2.2.3, where

$$|u'| \leq \frac{(p-1)C_3^p}{n(p-1)-2} \; r^{-1-\frac{2}{p-1}} = Cr^{-\frac{p+1}{p-1}} = \mathcal{O}(r^{-\frac{p+1}{p-1}}).$$

and

$$\int_0^\infty |u'|^{q-1} \mathrm{d}s \leq C \int_0^\infty s^{-1-\delta} \mathrm{d}s < \infty.$$

ii. Assume $p \leq p_F$. We choose $k > \frac{2}{p-1} - 1$ and multiply (2.3) by r^k and integrate from zero to r to obtain

$$\begin{aligned} |u'|r^k &= \int_0^r u^p s^k \mathrm{d}s - \int_0^r |u'|^q s^k \mathrm{d}s - (k-n+1)\int_0^r u' s^{k-1}\mathrm{d}s \\ &\leq \int_0^r u^p s^k \mathrm{d}s - (k-n+1)ur^{k-1} + (k-n+1)(k-1)\int_0^r us^{k-2}\mathrm{d}s \\ &= \mathcal{O}(r^{k+1-\frac{2p}{p-1}}). \end{aligned}$$

Since $k > \frac{2}{p-1} - 1$ and $u = \mathcal{O}(r^{\frac{-2}{p-1}})$, we have (2.15).
To prove (2.16), define $\delta = \frac{q(p+1)-2p}{p-1}$ as in Theorem 2.2.3, so that $\delta > 0$. By (2.15), we have $|u'|^{q-1} \leq Cr^{-1-\delta}$, hence

$$\int_0^\infty |u'|^{q-1}\mathrm{d}s \leq C\int_0^\infty s^{-1-\delta}\mathrm{d}s < \infty.$$

□

As a consequence of Theorem 2.3.2, we obtain a lower asymptotic estimate for ground states, which we obtained in the next corollary.

Corollary 2.3.3. *Suppose $p > 1$ and $q > 2p/(p+1)$. Then for any $\epsilon > 0$ there exists a constant $\rho > 0$ such that*

$$u \geq cr^{2-n-\epsilon}, \qquad r \geq \rho, \tag{2.17}$$

where c is a positive constant.

Proof. Define $\delta = \frac{q(p+1)-2p}{p-1}$ as in Theorem 2.2.3, so that $\delta > 0$. By (2.15) we have $|u'|^{q-1} \leq Cr^{-1-\delta}$. Therefore, given $\varepsilon > 0$, we can suppose without loss of generality that $u^p \geq 0$ for $r > \rho$ and ρ is large enough so that

$$u'' + \frac{n-1+\varepsilon}{r}u' + u^p \leq 0, \qquad r > \rho,$$

Hence in turn

$$(r^{n-1+\varepsilon}u')' \leq 0, \qquad r > \rho,$$

and $r^{n-1+\varepsilon}u'$ is decreasing function. Since u' obviously cannot be everywhere non-negative, it follows that

$$r^{n-1+\varepsilon}u' \to \quad \text{negative limit (possibly} -\infty)$$

as $r \to \infty$, hence $u' < 0$ for all suitably large r.
in particular for all large r, we have

$$r^{n-1+\varepsilon}u' \leq -C,$$

where C is some positive constant. Integrating this relation from any fixed value r to ∞ yields (2.17). □

2.4 Non-Existence of Ground States

Now, we are ready to prove the nonexistence of the radial ground states of the problem (2.1), that is, nonnegative, nontrivial radial solutions defined for all $r > 0$.

Theorem 2.4.1. *The equation (2.1) admits no positive radial ground states if*

$$2p/(p+1) < q < p, \qquad p \leq p_F.$$

Proof. i. $1 \leq p < p_F$ and $2p/(p+1) < q < p$.

The two estimates (2.17) and $(2.14)_1$ contradict each other when $1 \leq p < p_F$ and $\epsilon \leq 2p/(p-1) - n$.

ii. When $p = p_F$, then $q > (n+2)/(n+1)$.

By (2.15) and (2.16), then

$$u' = \mathcal{O}(r^{-(1+n)}),$$
$$\int_0^\infty |u'|^q s^{n+1} \mathrm{d}s \leq C \int_0^\infty |u'|^{q-1} \mathrm{d}s < \infty.$$

Multiplying (2.3) by r^{n+1} and integrating from zero to r to get

$$r^{n+1}|u'| = \int_0^r s^{n+1}u^p \mathrm{d}s - 2\int_0^r s^n u' \mathrm{d}s - \int_0^r s^{n+1}|u'|^q \mathrm{d}s \tag{2.18}$$

$$
\begin{aligned}
&= \int_0^r s^{n+1} u^p \mathrm{d}s - 2r^n u + 2n \int_0^r s^{n-1} u \mathrm{d}s - \int_0^r s^{n+1} |u'|^q \mathrm{d}s \\
&\leq \int_0^r s^{n+1} u^p \mathrm{d}s + 2n \int_0^r s^{n-1} u \mathrm{d}s - \int_0^r s^{n+1} |u'|^q \mathrm{d}s.
\end{aligned}
$$

Because the third integral is convergent, $|u'|r^{n+1}$ has a finite limit. Indeed the limit must be zero. For otherwise ur^n tends to a limit different from zero by L'Hospitals's rule, and $|u'|r^{n+1}$ tends to infinity by (2.18), which is impossible. If we now integrate from r to ∞, then

$$
\begin{aligned}
|u'|r^{n+1} &= -\int_r^\infty u^p s^{n+1} \mathrm{d}s - 2\int_r^\infty |u'| r^n \mathrm{d}s + \int_r^\infty |u'|^q s^{n+1} \mathrm{d}s \\
&\leq \int_r^\infty |u'|^q s^{n+1} \mathrm{d}s \\
&\leq C \int_r^\infty s^{-q(\frac{p+1}{p-1})} s^{n+1} \mathrm{d}s \\
&= C \int_r^\infty s^{-q(n+1)+(n+1)} \mathrm{d}s \\
&= \mathcal{O}(r^{1-(n+1)(q-1)}).
\end{aligned}
$$

Since $(q-1)(n+1) > 1$, therefore

$$
u = \mathcal{O}(r^{2-q(n+1)}).
$$

But this contradicts (2.17) for any $\epsilon > 0$ which is such that $\epsilon \leq q(n+1) - n$.

□

Remark 2.4.2. *In the first case of Theorem 2.4.1, i.e., when* $1 < p < p_F$ *and* $q > 2p/(p+1)$*, the equation (2.1) does not even admit singular radial ground states, which is nonnegative solutions of (2.1) on* $\mathbb{R}^n \setminus \{0\}$ *that tends to infinity at the origin. Indeed the proof only depends on the asymptotic behaviour of the solutions at infinity, having nothing to do with the behaviour at the origin.*

Chapter 3

The Blow-up Behaviour of the Positive Solutions

The semilinear parabolic problems which involve a dissipative gradient term were introduced for the first time by Chipot and Weissler in [**11**]. The main issue in their work was to extend Levine's work (see [**35**]) for the problem without gradient term to semilinear problems with gradient term, and more particularly, their objective was to investigate the possible effect of the gradient term on global existence or nonexistence of solutions.

The main purpose of this chapter is to show the important effect of the gradient term on the blow-up rate estimate when the power of the gradient term can increase to the power function of the solution. This chapter is organised as follows. In section 3.2 and 3.3 we extend the results of Chlebik, Fila and Quittner [**13**] for the problem of the heat equation with a dissipative gradient term defined in a ball and in a convex domain, respectively, considering the blow-up set in section 3.3 and observe that the gradient term can have opposite effects on the blow-up rate estimate in both domains: when the perturbation is strong ($2p/(p+1) < q < p$), the estimate is more singular than when the perturbation is weak ($q < 2p/(p+1)$), which is less singular.

3.1 Introduction

In this chapter we study the problem

$$\left.\begin{aligned} u_t &= \Delta u - h(|\nabla u|) + f(u) && \text{in } \ \Omega \times (0,T) \\ u &= 0 && \text{on } \ \partial\Omega \times (0,T) \\ u(x,0) &= u_0(x) && \text{in } \ \Omega, \end{aligned}\right\} \tag{3.1}$$

where

$$f \in C^1([0,\infty)), \quad f(u) > 0 \quad u > 0,$$

$$h \in C^1([0,\infty)), \qquad h(v) > 0, \quad h^{'}(v) \geq 0 \quad \text{for} \quad v > 0.$$

and

$$\tilde{h}(v) := vh^{'}(v) - h(v) \leq kv^q \ \text{ for } \ v > 0 \ \text{ and some } \ 0 \leq k < \infty, q > 1. \tag{3.2}$$

The Chipot-Weissler equation with Dirichlet boundary conditions and initial data is a special case of the problem (3.1), when $f(u) = u^p$ and $h(|\nabla u|) = \mu|\nabla u|^q$, namely

$$\left.\begin{aligned} u_t &= \Delta u - \mu|\nabla u|^q + u^p && \text{in } \ \Omega \times (0,T) \\ u(x,t) &= 0 && \text{on } \ \partial\Omega \times (0,T) \\ u(x,0) &= u_0(x) && \text{in } \ \Omega. \end{aligned}\right\} \tag{3.3}$$

The problem (3.3) was first introduced in [**11**] in order to investigate the possible effect of a gradient term on global existence or nonexistence of solutions of the problem (3.3). Moreover, the problem was studied later by many authors; they considered and investigated the existence of global and nonglobal positive solutions to (3.1), especially for $\mu > 0$ (see for instance [**16**],[**45**]). In particular, it is known from [**54**] that the finite time blow-up may occur for large initial data when $p > q$, on the other hand all solutions are global and bounded if $q \geq p$ if Ω is bounded (see [**16**],[**46**]). Furthermore, it has been shown in [**54**] that blow-up in finite time occurs when $p > q$ in $\mathbb{R}^n$, and also unbounded solutions always exist for the problem (3.3) when $p > q$ in whole space (see [**55**]). Therefore, it can be understood that there is a competition between the reaction term u^p, which may cause blow-up as in the problem

$$\left.\begin{aligned} u_t - \Delta u = u^p, \qquad & x \in \Omega, \quad t > 0, \\ u = 0, \qquad & x \in \partial\Omega, \quad t > 0, \\ u(x,0) = u_0(x), \qquad & x \in \Omega, \end{aligned}\right\} \tag{3.4}$$

and the gradient term, which fights against the blow-up.

Besides, the results are known for blow-up behaviour of nonglobal solutions of (3.3) near blow-up. For instance in [**53**] the self-similar solution is considered when $q = 2p/(p+1)$, and it has been proven that u blows up at the single point $x = 0$ and admits a limiting profile given by

$$u(x,T) = C|x|^{-2/(p-1)}, \quad \text{for all} \;\; x \neq 0.$$

Moreover, in [**13**] the authors showed that if $\Omega = B_R = \{x \in \mathbb{R} : \; |x| < R\}$ and u_0 is radial nonincreasing, then the gradient term has a strong effect on the final blow-up profile of the solution of (3.3) when $2p/(p+1) < q < p$ and it takes the form

$$u(r,t) \leq Cr^{-\alpha}, \qquad (r,t) \in (0,R] \times [0,T) \tag{3.5}$$

for any $\alpha > q/(p-q)$, while there is no difference between the final blow-up profile of the problem (3.3) and (3.4) when $q \in (1, 2p/(p+1))$, since in both problems, the final blow-up profile takes the form (3.5) for any $\alpha > 2/(p-1)$. Also, it was proved that $x = 0$ is the only possible blow-up point in the radial case, and the blow-up set is a compact subset of Ω if Ω is convex and $q < 2p/(p+1)$, which means that the blow up points will not occur on the boundary (see [**13**]).

With regard to the estimate of the blow-up rates of the problem (3.3), it has been proved in [**13, 12, 18, 58**] in the case $q < 2p/(p+1)$ and $u \geq 0$ that the upper (lower) blow-up estimate takes a form similar to that of the equation (3.4), is namely

$$C_1(T-t)^{-1/(p-1)} \leq \|u(t)\|_\infty \leq C_2(T-t)^{-1/(p-1)}, \qquad \text{as} \;\; t \to T. \tag{3.6}$$

The estimates (3.6) hold under the following conditions:

- when $\Omega = \mathbb{R}^n$, $p \leq 1 + 2/n$ [**12**];
- when $\Omega = \mathbb{R}^n$ or $\Omega = B_R$, $p < (n+2)/(n-2)$, u is nonincreasing radially symmetric and $u_t \geq 0$ [**58**] ;
- when Ω is bounded convex and ($u_t \geq 0$ or $p \leq 1 + 2/n$) [**13**];
- when Ω is arbitrary and $p \leq 1 + 2/(n+1)$ [**18**].

In this chapter our aim is to prove that the blow-up set of the problem (3.1) is a compact subset of Ω if Ω is convex and $q \in (\frac{2p}{p+1}, p)$, and also to show that the gradient term has remarkable influence on the upper blow up rate estimate of the problem (3.3) when $q \in (\frac{2p}{p+1}, p)$, which is different from the upper rate estimate of (3.4) (see (3.6)) where $q \in (1, \frac{2p}{p+1})$.

3.2 The Radially Symmetric Case

The following lemma is a modification of the argument in the proof of Lemma 2.1 in [**22**], which guarantees that $u_r < 0$ in $\Omega \cap \{r > 0\}$ and $u_{rr}(0,t) < 0$, for $t \in (0,T)$.

Lemma 3.2.1. *Consider problem (3.3) with $1 < q < p$, $\mu > 0$, and $\Omega = B_R$. Let $u_0 \in X_+$, be radial non-increasing, and assume $T := T_{max}(u_0) < \infty$. Then*

$$u_r < 0 \quad \textit{for all} \quad r \in (0,R], \; t \in (0,T)$$

and

$$u_{rr}(0,t) < 0 \quad for \quad t \in (0,T).$$

Proof. Denote $\Omega_1 := \Omega \cap \{x : \; x_1 > 0\}$. We notice that $v := u_{x_1}$ satisfies

$$v_t - \Delta v = f^{'}(u)v - (h^{'}(|\nabla u|)\frac{\nabla u}{|\nabla u|}) \cdot \nabla v \quad \text{in} \quad \Omega_1,$$

since $v = 0$ for $x \in \partial\Omega_1$, $x_1 = 0$, and $v < 0$ for $x \in \partial\Omega_1$, $x_1 > 0$. Then the maximum principle implies $v < 0$ for $x \in \Omega_1$ and $v_{x_1}(0,t) = u_{x_1 x_1}(0,t) < 0$ for $t > 0$. Hence $u_{rr}(0,t) < 0$. $\square$

Under an additional assumption of monotonicity in time, it can be seen in the proof of Theorem 39.2 in [**47**] that there is a simple property of gradient solution of the problem (3.3), which we shall show in the next theorem.

Theorem 3.2.2. *Consider the problem (3.3) with* $1 < q < p$, $\mu > 0$, *and* $\Omega = B_R$. *Let* $u_0 \in X_+$, *be radial non-increasing,* $u_t \geq 0$ *in* Q_T, *and assume* $T := T_{max}(u_0) < \infty$. *Then there exists* $C > 0$ *such that*

$$\|u_r(t)\|_\infty \leq C_1 u^\gamma(0,t), \tag{3.7}$$

with $\gamma = \min\big((p+1)/2, p/q\big) > 1$.

Proof. Since $u_t \geq 0$ and $u_r \leq 0$, we have

$$\frac{\partial}{\partial r}\Big(\frac{1}{2}u_r^2 + \frac{1}{p+1}u^{p+1}\Big) = (u_{rr} + u^p)u_r = \Big(u_t + \mu|u_r|^q - \frac{n-1}{r}u_r\Big)u_r \leq 0,$$

hence by integration with respect to r, we have

$$\Big(\frac{1}{2}u_r^2 + \frac{1}{p+1}u^{p+1}\Big)(r,t) \leq \frac{1}{p+1}u^{p+1}(0,t).$$

Therefore, we get (3.7) with $\gamma = (p+1)/2$, $C_1 = C_1(p)$. However, for every $t \in (0,T)$, at a point $r \in (0,R]$ where $|u_r(.,t)|$ achieves its maximum, we have

$$\mu|u_r|^q = u^p + u_{rr} - u_t + \frac{n-1}{r}u_r \leq u^p,$$

due to $u_t \geq 0$, $u_r \leq 0$ and $u_{rr}(r,t) \leq 0$. This gives (3.7) with $\gamma = p/q$, and $C_1 = \mu^{-1/q}$. □

3.2.1 Blow-up Rate Estimate

In order to derive a formula to the rate estimate for problem (3.3), we need first to recall the following theorem in [**13**], which shows that $r = 0$ is the only blow up point and the behaviour of solutions at blow-up is different for equations (3.4) and (3.3) with $q \in (2p/(p+1), p)$.

Theorem 3.2.3. [**13**] *Consider problem (3.3) under assumptions of Lemma 3.2.1. Then 0 is the only blow-up point. Moreover, for all $\alpha > \alpha_0$, it holds*

$$u(r,t) \le C_\alpha r^{-\alpha}, \quad 0 \le t < T, \quad 0 < r \le R,$$

with

$$\alpha_0 = \begin{cases} 2/(p-1) & \text{if } 1 < q \le 2p/(p+1), \\ q/(p-q) & \text{if } 2p/(p+1) < q < p. \end{cases}$$

We are ready now to derive a formula for the rate estimate for the blow-up solutions of problem (3.3).

Theorem 3.2.4. *Let $\Omega = B_R$, u be a blow-up solution to problem (3.1), where $u_0 \in X_+ \cap C^2(\bar{B}_R)$, is radial non-increasing and satisfies the monotonicity assumption*

$$\Delta u_0 + f(u_0) - h(|\nabla u_0|) \ge 0 \qquad \text{in} \quad B_R.$$

Let $f(u) = u^p$ with $p > 1$, assume h satisfies (3.2) with some $q \in (2p/(p+1), p)$. Then there exits $C > 0$ such that the upper blow-up rate estimate takes the following form

$$u(0,t) \le C(T-t)^{\frac{-q}{2(p-q)}},$$

where T is the blow-up time.

Proof. We introduce the function

$$J = u_t - \delta F(u), \quad (x,t) \in B_R \times (0,T),$$

where $\delta > 0$ and F is a non-negative function to be determined, $F^{'} \ge 0$, $F'' \ge 0$. Since $u_t = \delta F + J$,

$$\begin{aligned} J_t - \Delta J &= u_{tt} - \Delta u_t - \delta F'[u_t - \Delta u] + \delta F''|\nabla u|^2 \\ &= f'[J + \delta F] - h'.\frac{\nabla u}{|\nabla u|}.(\nabla J + \delta F'.\nabla u) - \delta F'[f - h] + \delta F''|\nabla u|^2 \\ &= f'J - h'.\frac{\nabla u}{|\nabla u|}.\nabla J + \delta f'F - \delta F'[h'.|\nabla u| - h] - \delta F'f + \delta F''|\nabla u|^2. \end{aligned}$$

Our aim is to prove

$$f'F - F'f + F''|\nabla u|^2 - F'[h'.|\nabla u| - h] \geq 0. \tag{3.8}$$

It will then follow that J cannot attain a negative minimum in $\Omega \times (0,t]$ for any $t < T$. From (3.7)

$$\|\nabla u\|_\infty \leq C u^\gamma(0,t), \quad x \in B_R, \quad \text{with} \quad \gamma = \min\Big\{\frac{p+1}{2}, \frac{p}{q}\Big\}.$$

Since $q > \frac{2p}{p+1}$ then $\frac{2p}{q} - 1 < p$, hence $\gamma = \frac{2p}{q} - 1$. We choose $F(u) = u^{\frac{2p}{q}-1}$. Then from (3.2) and $u_r \leq 0$ (u is a non-increasing function) for $0 < t < T$ and $0 < r \leq R$ there is $C > 1$ such that $u(0,t) = Cu(r,t)$

$$\begin{aligned}
f'F - F'f + F''|\nabla u|^2 - F'[h'.|\nabla u| - h] &\geq f'F - F'f + F''|\nabla u|^2 - kF'|\nabla u|^q \\
&= \Big(p - \frac{2p}{q} + 1\Big) u^{p+\frac{2p}{q}-2}(r,t) + \Big(\frac{2p}{q} - 1\Big)\Big(\frac{2p}{q} - 2\Big) u^{\frac{2p}{q}-3}(r,t)|\nabla u(r,t)|^2 \\
&\qquad - kC_1\Big(\frac{2p}{q} - 1\Big) u^{p+\frac{2p}{q}-2}(0,t) \\
&= \Big(p - \frac{2p}{q} + 1\Big) u^{p+\frac{2p}{q}-2}(r,t) + \Big(\frac{2p}{q} - 1\Big)\Big(\frac{2p}{q} - 2\Big) u^{\frac{2p}{q}-3}(r,t)|\nabla u(r,t)|^2 \\
&\qquad - kC_2\Big(\frac{2p}{q} - 1\Big) u^{p+\frac{2p}{q}-2}(r,t) \geq 0
\end{aligned}$$

for $k > 0$ small enough and $k \leq C_3\Big(\frac{q(p+1)-2p}{2p-q}\Big)$, $C_3 = C_2^{-1}$. By Theorem 3.2.3, u blows up in finite time at $r = 0$. Therefore, if $\epsilon > 0$ is small enough then

$$F(u) \leq C_0 < \infty \quad \text{if} \;\; x \in \partial B_\epsilon, \;\; 0 < t < T.$$

Then by applying the maximum principle to u_t we also have $u_t \geq c > 0$ on the parabolic boundary of $B_\epsilon \times (\epsilon, T)$. It follows that $J > 0$ on the parabolic boundary of $B_\epsilon \times (\epsilon, T)$, which leads to

$$u_t(0,t) \geq \delta F(u(0,t)), \qquad t \in (\epsilon, T). \tag{3.9}$$

Let $G(s) = \int_s^\infty \frac{\mathrm{d}u}{F(u)}$. Then (3.9) implies that

$$-\frac{\mathrm{d}G(t)}{\mathrm{d}t} = \frac{u_t}{F(u)} \geq \delta.$$

Hence by integration

$$G(u(0,t)) \geq G(u(0,t)) - G(u(0,T)) \geq \delta(T-t).$$

Therefore also

$$u(0,t) \leq G^{-1}(\delta(T-t)), \qquad t \in (\epsilon, T). \tag{3.10}$$

Since $f(u) = u^p$ with $p > 1$, we can choose $F(u) = u^{\frac{2p}{q}-1}$, hence (3.8) will be satisfied. Then

$$u(0,t) \leq C(T-t)^{\frac{-q}{2(p-q)}}.$$

□

3.3 The Convex Case

We need the simple properties of the time-derivative u_t and the Laplacian of u to the problem (3.1) if Ω is convex, which we will show in the next lemma

Lemma 3.3.1. *Consider the problem (3.1-3.2), let Ω be a convex bounded domain. If $1 < q < p$, then there is $\Omega' \subseteq \Omega$ such that*

i. $u_t \geq C$ in Ω' if $\Delta u_0 + f(u_0) - h(|\nabla u_0|) \geq 0$ in Ω.

ii. $\Delta u(x,t) < 0$ in Ω'.

Proof. Take any point $y_0 \in \partial\Omega$. Now, let new orthogonal coordinates be chosen in such a way that y_0 is the origin and $(1, 0, ..., 0)$ is the outward normal at y_0.
Let $\Omega' := \Omega \cap \{x \in \mathbb{R}^n : x_1 > a\}$, where $a < 0$.

i. Let $w = u_t$.

Then

$$\left.\begin{aligned} &w_t - \Delta w = f'(u)w - h'(|\nabla u|)\frac{\nabla u}{|\nabla u|} \cdot \nabla w && \text{in } \Omega \times (0,T) \\ &w(x,t) = 0 && \text{on } \partial\Omega \times (0,T) \\ &w(x,0) = w_0(x) \geq 0 && \text{in } \Omega. \end{aligned}\right\} \tag{3.11}$$

From (3.11) and the maximum principle, there exists C such that $w = u_t \geq C$.

ii. Let $v = u_{x_1}$.

We notice that v satisfies

$$v_t - \Delta v = f'(u)v - h'(|\nabla u|)\frac{\nabla u}{|\nabla u|} \cdot \nabla v \quad \text{in} \;\; \Omega'.$$

Since $v = 0$ for $x \in \partial\Omega'$, $x_1 = a$, and $v < 0$ for $x \in \partial\Omega'$, $x_1 > a$, the maximum principle implies $v < 0$ for $x \in \Omega'$ and $v_{x_1}((a,0),t) = u_{x_1x_1}((a,0),t) < 0$ for $t > 0$. Hence $\Delta u((a,0),t) < 0$. In the above proof it can be shown that a can be chosen independently of the initial point $y_0 \in \partial\Omega$. Hence, by varying y_0 along $\partial\Omega$ we conclude that there is a subset Ω' of Ω such that $\Delta u(x,t) < 0$ in $\Omega' \subseteq \Omega$.

□

The following theorem shows simple properties of the first order spatial derivative to the problem (3.3).

Theorem 3.3.2. *Consider the problem (3.3) with $1 < q < p$, $\mu > 0$, and Ω is a convex bounded domain, and assume in addition that $u_t \geq 0$ in Q_T. Then there is $\Omega' \subseteq \Omega$ such that*

$$\|\nabla u(x,t)\|_\infty \leq C_1 u^\gamma(a,t), \qquad (x,t) \in \Omega' \times (0,T), \tag{3.12}$$

where $\gamma = \min(\frac{p+1}{2}, \frac{p}{q}) > 1$.

Proof. Let Ω' be defined as previously in Lemma 3.3.1. Since $u_t \geq 0$, and $u_{x_1} < 0$ in $\Omega' \subseteq \Omega$ as we explained in Lemma 3.3.1, we have

$$\nabla(\frac{1}{2}(\nabla u)^2 + \frac{1}{p+1}u^{p+1}) = \nabla u \Delta u + u^p \nabla u = (\Delta u + u^p)\nabla u = (u_t + \mu|\nabla u|^q).\nabla u \leq 0,$$

hence, in $\Omega' \times (0,T)$ at any (x,t) such that $x = (x_1, 0)$, $a \leq x_1 < 0$, and by integrating with respect to x_1 we get

$$(\frac{1}{2}(\nabla u)^2 + \frac{1}{p+1}u^{p+1})((x_1,0),t) \leq \frac{1}{p+1}u^{p+1}((a,0),t),$$

hence

$$\|\nabla u(t)\|_\infty \leq C_1 u^{\frac{p+1}{2}}((a,0),t), \quad C_1 = C_1(p).$$

Therefore, we get (3.12) with $\gamma = (p+1)/2$.

On the other hand, for each $t \in (0,T)$, at a point $x = (x_1, 0) \in \Omega'$, $a \leq x_1 < 0$ where $|\nabla u(.,t)|$ achieves its maximum, we have

$$\mu|\nabla u|^q = u^p + \Delta u - u_t \leq u^p,$$

due to $u_t \geq 0$, $\nabla u \leq 0$ and $\Delta u(x,t) \leq 0$. This yields (3.12) with $\gamma = p/q$ and $C_1 = \mu^{-1/q}$.

Hence, since a is chosen independently of the initial point $y_0 \in \partial\Omega$, and by varying y_0 along $\partial\Omega$ we obtain the claim (3.12). □

3.3.1 Blow-up Set

We shall prove in the next theorem that the blow-up in the problem (3.1) takes place away from the boundary if Ω is convex and $q > 2p/(p+1)$. In order to prove that, we need first to introduce the following lemma.

Lemma 3.3.3. *If $\frac{2p}{p+1} < q < p$, $\gamma = \frac{p}{q}$ and $k \in (0,\infty)$, then for $|a|$ small enough, there holds*

$$\frac{1}{2}[(p-\gamma)u^{p+\gamma-1} + 4|a|^{-2}u^\gamma] \geq 4\epsilon\gamma|a|u^{2\gamma-1}, \tag{3.13}$$

for every $u \geq 0$.

Proof. Young's inequality

$$\frac{A^\alpha}{\alpha} + \frac{B^\beta}{\beta} \geq AB, \quad A, B \geq 0, \quad \alpha, \beta > 1, \quad \frac{1}{\alpha} + \frac{1}{\beta} = 1,$$

with the choice $\gamma = \frac{p}{q}$,

$$\alpha = \frac{[q(p+1) - 2p] + 2(p-q)}{p-q}, \quad \beta = \frac{q(p-1)}{p(q-1)},$$
$$\frac{A^\alpha}{\alpha} = \frac{1}{2}(p-\gamma)u^{p+\gamma-1}, \quad \frac{B^\beta}{\beta} = 2|a|^{-2}u^\gamma,$$

which implies the claim. □

Theorem 3.3.4. *Let Ω be a convex bounded domain. If $f(u) = u^p$, $p > 1$ and h satisfies (3.2) with some $q \in (\frac{2p}{p+1}, p)$, then the blow-up set of any solution of (3.1-3.2) is a compact subset of Ω.*

Proof. We assume without loss of generality that

$$\frac{\partial u_0}{\partial \nu} < 0 \qquad \text{on} \quad \partial\Omega, \tag{3.14}$$

since ν is unit normal at any point $x \in \partial\Omega$.
We take any point $y_0 \in \partial\Omega$. Now, let the new orthogonal coordinates be chosen in such a way that y_0 is the origin and $(1, 0, ..., 0)$ is the outward normal at y_0.
Let $\Omega_a^+ = \Omega \cap \{x \in \mathbb{R}^n : x_1 > a\}$, where $a < 0$.
Using standard reflection principle we easily conclude from (3.14) that

$$u_{x_1} < 0 \qquad \text{in } \Omega_a^+ \times (0, T), \tag{3.15}$$

provided $|a|$ is small enough. To obtain an estimate from below on $-u_{x_1}$ in $\Omega_a^+ \times (0, T)$, we introduce a function

$$J = u_{x_1} + c(x_1)F(u). \tag{3.16}$$

in $\Omega_a^+ \times (0, T)$, where $c,\ F$ are nonnegative functions to be determined, and $c' \geq 0$, $F' \geq 0$, $F'' \geq 0$.
We compute that

$$\begin{aligned} J_t - \Delta J + h'.\frac{\nabla u}{|\nabla u|}.\nabla J - (f' + c'h'F.\frac{1}{|\nabla u|} - 2c'F')J \\ = cF'f - cFf' + cF'\tilde{h}(|\nabla u|) - cc'h'F^2.\frac{1}{|\nabla u|} + 2cc'FF' \\ - c''F - c(J - cF)^2F''. \end{aligned}$$

Using the relation

$$(J - cF)^2 = c^2F^2 + (J - 2cF)J,$$

$$\tilde{h}(|\nabla u|) = \tilde{h}(-\nabla u) = \tilde{h}(cF - J) \leq k(cF - J)^q < kc^qF^q + |J|^q.$$

Then

$$J_t - \Delta J + h'.\frac{\nabla u}{|\nabla u|}.\nabla J - bJ \leq c\{F'f - f'F + kc^q F'F^q - c'h'F^2.\frac{1}{|\nabla u|} + 2c'FF' - c^2F^2F'' - \frac{c''}{c}F\}, \quad (3.17)$$

$$\text{where} \quad b = f' + c'h'F.\frac{1}{|\nabla u|} - 2c'F' + cF'|J|^{q-1} - c(|J| - 2cF)F''.$$

If F and c satisfy

$$f'F - F'f - 2c'F'F - kc^qF'F^q + \frac{c''}{c}F + c^2F^2F'' \geq 0, \quad (3.18)$$

then the right hand side in (3.17) is non-positive. Therefore, J cannot attain a positive maximum in $\Omega_a^+ \times (0, t]$ for every $t < T$.

Next, we show that (3.18) is satisfied for

$$c(x_1) = \epsilon(x_1 - a)^2,$$

with $|a|$ small enough and some suitably chosen F.

Recall $f(u) = u^p$, $p > 1$, choosing $F(u) = u^\gamma$, $1 < \gamma < p$, it is sufficient to prove

$$(p-\gamma)u^{p+\gamma-1} - 4\epsilon\gamma|a|u^{2\gamma-1} - \epsilon^q k\gamma|a|^{2q}u^{q\gamma+\gamma-1} + 2|a|^{-2}u^\gamma + \epsilon^2|a|^2\gamma(\gamma-1)u^{3\gamma-2} \geq 0. \quad (3.19)$$

For every $u \geq 0$ and if $|a|$ is small enough, we can prove that

$$\frac{1}{2}(p-\gamma)u^{p+\gamma-1} \geq \epsilon^q k\gamma|a|^{2q}u^{\gamma q+\gamma-1} \quad (3.20)$$

if we choose $\gamma = p/q$ and $\epsilon < (\frac{p-\gamma}{2k\gamma|a|^{2q}})^{\frac{1}{q}}$.

Inequality (3.19) follows immediately from (3.20) and (3.13).

Next, observe that $J < 0$ on $\{x : x_1 > a\}$ by (3.15) and $J < 0$ on $\{t = 0\}$ by (3.14). The maximum principle yields also $J < 0$ on $\Gamma \times (0, T)$, where $\Gamma = \partial\Omega \cap \{x : x_1 > a\}$. Hence, $J < 0$ in $\Omega_a^+ \times (0, T)$.

Consequently

$$-u_{x_1} \geq \epsilon(x_1 - a)^2 F(u)$$

at any (x,t) such that $x = (x_1, 0)$, $a \le x_1 < 0$.

Integrating with respect to x_1 and denoting

$$G(s) = \int_s^\infty \frac{\mathrm{d}u}{F(u)},$$

we get

$$G[u((x_1,0),t),t] \ge \int_a^{x_1} c(\rho)\mathrm{d}\rho = \frac{1}{3}(x_1 - a)^3$$

and therefore,

$$u((x_1,0),t) \le G^{-1}[\frac{1}{3}(x_1 - a)^3]$$

Thus, u is uniformly bounded on

$$\{(x_1,0) : x_1 \in [\frac{a}{2}, 0]\} \times (0,T).$$

Then we can chose a independently of the initial point $y_0 \in \partial\Omega$. Hence, by varying y_0 along $\partial\Omega$ we conclude that there is neighbourhood Ω' of $\partial\Omega$ such that u is uniformly bounded in $\Omega' \times (0,T)$. $\square$

3.3.2 Blow-up Rate Estimate

In this subsection we consider the upper blow-up rate estimates for the solutions of problem (3.1) with (3.2).

Theorem 3.3.5. *Let Ω be a convex bounded domain and let $u_0 \in X_+ \cap C^2(\bar{\Omega})$ be such that*

$$\Delta u_0 + f(u_0) - h(|\nabla u_0|) \ge 0 \qquad in \ \ \Omega.$$

If $f(u) = u^p$ with $p > 1$ and h satisfies (3.2) with some $q \in (\frac{2p}{p+1}, p)$, then any solution u of (3.1) which blows up at $t = T$ satisfies

$$u(x,t) \le C(T-t)^{\frac{-q}{2(p-q)}}. \tag{3.21}$$

Proof. For any $\eta > 0$ that is small enough, set

$$\Omega^\eta = \{x \in \Omega : \mathrm{dist}(x, \partial\Omega) > \eta\}.$$

We shall derive a lower bound on u_t away from the parabolic boundary of $\Omega \times (0, T)$. We introduce the function

$$J = u_t - \delta F(u),$$

where $\delta > 0$ and F is a non-negative function to be determined, $F^{'} \geq 0$, $F^{''} \geq 0$. Since $u_t = \delta F + J$, by the proof of Theorem 3.2.4, a direct calculation shows

$$J_t - \Delta J = f^{'} J - h^{'} \frac{\nabla u}{|\nabla u|} \cdot \nabla J + \delta f^{'} F - \delta F^{'}[h^{'}|\nabla u| - h] - \delta F^{'} f + \delta F^{''}|\nabla u|^2.$$

and we have to prove that

$$f^{'} F - F^{'} f + F^{''}|\nabla u|^2 - F^{'}[h^{'}|\nabla u| - h] \geq 0. \tag{3.22}$$

From (3.12)

$$\|\nabla u\|_\infty \leq C u^\gamma(x,t), \quad x \in \Omega^\eta \;\; \text{with} \;\; \gamma = \min\left\{\frac{p+1}{2}, \frac{p}{q}\right\}$$

and since $q > \frac{2p}{p+1}$ then $\frac{2p}{q} - 1 < p$, hence $\gamma = \frac{2p}{q} - 1$. We choose $F(u) = u^{\frac{2p}{q}-1}$, then from (3.1) and arguing as in the proof of Lemma 3.3.1, $u_{x_1} < 0$ for $x_1 > a$, we obtain for any (x,t) such that $x = (x_1, 0)$, $a \leq x_1 < 0$ that, $u(x,t) = Cu(a,t)$, $C > 1$,

$$f^{'} F - F^{'} f + F^{''}|\nabla u|^2 - F^{'}[h^{'}.|\nabla u| - h] \geq f^{'} F - F^{'} f + F^{''}|\nabla u|^2 - kF^{'}|\nabla u|^q$$

$$\begin{aligned}
&= \Big(p - \frac{2p}{q} + 1\Big) u^{p+\frac{2p}{q}-2}(x,t) + \Big(\frac{2p}{q} - 1\Big)\Big(\frac{2p}{q} - 2\Big) u^{\frac{2p}{q}-3}(x,t)|\nabla u(x,t)|^2 \\
&\qquad\qquad - kC_1\Big(\frac{2p}{q} - 1\Big) u^{p+\frac{2p}{q}-2}(a,t) \\
&= \Big(p - \frac{2p}{q} + 1\Big) u^{p+\frac{2p}{q}-2}(x,t) + \Big(\frac{2p}{q} - 1\Big)\Big(\frac{2p}{q} - 2\Big) u^{\frac{2p}{q}-3}(x,t)|\nabla u(x,t)|^2 \\
&\qquad\qquad - kC_2\Big(\frac{2p}{q} - 1\Big) u^{p+\frac{2p}{q}-2}(x,t) \geq 0
\end{aligned}$$

for $k > 0$ small enough and $k \le C_3\left(\frac{q(p+1)-2p}{2p-q}\right)$, $C_3 = C_2^{-1}$. We obtain that J cannot attain a negative minimum in $\Omega \times (0, t]$ for any $t < T$. By Theorem 3.3.4, the set of blow-up points is compact subset of Ω. Therefore, if $\eta > 0$ is small enough then $F(u) \le C_0 < \infty$ if $x \in \partial\Omega^\eta$, $0 < t < T$. By Lemma 3.3.1(i), $u_t \ge C > 0$ on the parabolic boundary of $\Omega^\eta \times (\eta, T)$ provided δ is chosen sufficiently small and consequently $J > 0$ in $\Omega^\eta \times (\eta, T)$.

Hence,

$$\frac{u_t}{F(u)} \ge \delta \qquad \text{in } \ \Omega^\eta \times (\eta, T). \tag{3.23}$$

Let $G(s) = \int_s^\infty \frac{\mathrm{d}u}{F(u)}$. Then (3.23) implies that

$$-\frac{\mathrm{d}G(t)}{\mathrm{d}t} = \frac{u_t}{F(u)} \ge \delta,$$

by integration

$$G(u(x,t)) \ge G(u(x,t)) - G(u(x,T)) \ge \delta(T-t).$$

Therefore,

$$u(x,t) \le G^{-1}(\delta(T-t)), \qquad (x,t) \in \Omega^\eta \times (\eta, T).$$

This gives an upper bound on the blow up rate as $t \to T$.

Since $f(u) = u^p$ with $p > 1$, we can choose

$$F(u) = u^{\frac{2p}{q}-1}$$

and (3.22) is satisfied. For this choice of F, we obtain the estimate

$$u(x,t) \le C(T-t)^{\frac{-q}{2(p-q)}}.$$

□

Chapter 4

Parabolic Liouville-type Theorems and the Universal Estimates

Parabolic Liouville-type theorems have important applications, they can be efficiently used in the proofs of a priori bounds, singularity and decay estimates, see [**44**]. Liouville-type theorems mean, the statement of nonexistence of nontrivial bounded solutions that are defined for all negative and positive times on the whole space.

The main aim of the chapter is to prove the Liouville-type theorems and to establish the universal initial and final blow-up rates for the parabolic problem $u_t - \Delta u = u^p - \mu|\nabla u|^q$ when $q > 2p/(p+1)$. In section 4.2 we derive criteria for initial data which guarantee occurring the blow-up in finite time. Section 4.3 is devoted to obtain the Liouville-type theorem under the restriction $p < p_F$ in the radial case and under the stronger restriction $p < p_B$ in the general (nonradial) case. Next we shall study in section four the (universal) a priori bound for global solutions and the usual blow-up rate estimates for $q = 2p/(p+1)$ and for $q > 2p/(p+1)$.

4.1 Introduction

In this chapter, we consider the parabolic equation

$$u_t - \Delta u = u^p - \mu|\nabla u|^q, \quad x \in \mathbb{R}^n,\ t \in \mathbb{R}, \tag{4.1}$$

for $p > 1$ and $p > q > 2p/(p+1)$.

It's well known that both elliptic and parabolic Liouville-type theorems play an important role in the study of a priori estimates and (blow-up) singularities. For example, it has been shown in [**25**] that the nonlinear elliptic problem

$$-\Delta u = u^p \qquad \text{in } \ \mathbb{R}^n$$

does not admit any positive solution if $p < p_S$. It was proved also that even positive supersolutions of the equation (4.1) cannot exist if $p < \frac{n}{n-2}$ (see Theorem 8.4 in [**47**]).

General nonexistence results for the problem (4.1) were obtained in [**4**]. It has been shown there that the differential inequality

$$-Qu \geq f(u) \quad \text{ in } \mathbb{R}^n \setminus B_{R_0},$$

where $n \geq 3$ and Q is a fully nonlinear operator, does not admit positive solutions provided that f is continuous and positive in $(0, \infty)$ and

$$\lim_{s \to 0^+} \inf f(s)/s^{\frac{n}{n-2}} > 0.$$

It has been shown in [**2**] that for the problem

$$-\Delta u + |\nabla u|^q = \lambda f(u) \quad \text{ in } \mathbb{R}^n \setminus B_{R_0}$$

positive supersolutions do not exist if $q > 1$ and the function f can be compared with a power p near zero or infinity.

Furthermore, Liouville-type theorems for nonnegative supersolutions of the elliptic problem

$$-\Delta u + b(x)|\nabla u| = c(x)u \quad \text{ in } \mathbb{R}^n \setminus B_{R_0} \tag{4.2}$$

were proved, where b and c are allowed to be unbounded. It was shown that if $\liminf_{x\to\infty} 4c(x) - b(x)^2 > 0$ then no positive supersolutions can exist. However, it is known previously in [**6**] and [**7**] that if $b, c \in C(\mathbb{R}^n)$, the problem

$$-\Delta u + b(x)\nabla u \geq c(x)u \qquad \text{in } \mathbb{R}^n$$

does not admit any positive solution provided that b and c are bounded and

$$\lim_{x\to\infty}\inf \left[4c(x) - |b(x)|^2\right] > 0. \tag{4.3}$$

Moreover, as a particular case of generalization of fully nonlinear elliptic operator, it has been obtained in [**49**] that, if b and c are bounded in $\mathbb{R}^n \setminus B_{R_0}$ and (4.3) holds, then the problem (4.2) does not admit positive supersolutions.

On the other hand, the parabolic Liouville type theorem for all $p < p_S$ for the heat equation

$$u_t - \Delta u = u^p, \qquad x \in \mathbb{R}^n, \quad t \in \mathbb{R}, \tag{4.4}$$

in the case of radial solutions was proved in [**43**]. The previous theorem is optimal, since it is known that for $n \geq 3$ and $p \geq p_S$, (4.4) admits positive stationary solutions which are radial and bounded. However, the Liouville type theorem for (4.4) in the general case under stronger restriction $p < p_B$ has been obtained in [**47**].

In [**44**] Liouville-type theorems for the problem (4.4) in half space $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n : x_1 > 0\}$ with boundary condition have been studied, if $n \leq 2$ or $p < p_B(n-1)$ and $n \geq 3$.

Furthermore, in Section 4.4, we will study problems involving first-order derivative

$$\left.\begin{aligned} u_t - \Delta u &= u^p + g(u, \nabla u), && x \in \Omega, \quad 0 < t < T \\ u &= 0, && x \in \partial\Omega, \quad 0 < t < T. \end{aligned}\right\} \tag{4.5}$$

The special case of the model problem (4.5) that we will also study in this section is

$$\left.\begin{aligned} u_t - \Delta u &= u^p - \mu|\nabla u|^q, && x \in \Omega, \quad 0 < t < T, \\ u(x,t) &= 0, && x \in \partial\Omega, \quad 0 < t < T. \end{aligned}\right\} \tag{4.6}$$

The (universal) a priori bound for global solution and the usual blow-up rate estimate of the perturbed problem (4.5) has been studied in [**47**] when the perturbation term is not too strong, it was proved that the universal bounds of the problem (4.5) when $g = 0$ and $p > 1$, remain valid for the perturbed problem (4.5) if $p > q > 1$ and $1 < q < 2p/(p+1)$, and these estimates take the form

$$u(x,t) + |\nabla u(x,t)|^{\frac{2}{p+1}} \leq C(1 + t^{\frac{-1}{p-1}} + (T-t)^{\frac{-1}{p-1}}), \quad x \in \Omega, \quad 0 < t < T. \tag{4.7}$$

Our purpose in this chapter is to present new Liouville-type theorems for parabolic equations with a gradient term if $q > 2p/(p+1)$ in radial case and in general case. Furthermore, we prove that the estimate (4.7) is true for the problem (4.5) for the value $q = 2p/(p+1)$. On the other hand, we show that, the universal bounds of the form (4.7) do not remain valid for the perturbed problem (4.6) if $q > 2p/(p+1)$, which takes the form

$$u(x,t) \leq C(p,\Omega)\big(1 + t^{-q/2(p-q)} + (T-t)^{-q/2(p-q)}\big).$$

4.2 Blow-up

In this section we consider the model problem

$$\left.\begin{aligned} u_t - \Delta u = f(u) - h(|\nabla u|), &\qquad x \in \Omega, \quad t > 0,\\ u = 0, &\qquad x \in \partial\Omega, \quad t > 0,\\ u(x,0) = u_0(x), &\qquad x \in \Omega. \end{aligned}\right\} \tag{4.8}$$

We are going to give here a criterion for u_0, which guarantees that blow-up occurs if one starts above a positive equilibrium. In order to prove that, we need to prepare the following lemma.

Lemma 4.2.1. *Assume Ω bounded and consider problem (4.8) where $f : \mathbb{R} \to \mathbb{R}$ is a convex C^1-function with $f(0) = 0$. Let u_0, $\underline{u_0} \in L^\infty(\Omega)$ be such that $u_0 \geq \underline{u_0}$, $u_0 \not\equiv \underline{u_0}$. Let $u, \underline{u}$ be the corresponding solutions of (4.8), and fix $\tau \in (0, T_{max}(u_0))$.*

Then $T_{max}(\underline{u_0}) \geq T_{max}(u_0)$, *and if* $f(u) = u^p$, $h(|\nabla u|) = |\nabla u|^q$, $p > q$, *then there exists* $\alpha > 1$ *such that*

$$u \geq \alpha \underline{u}, \qquad \tau \leq t < T_{max}(u_0).$$

Proof. Since $u_0 \geq \underline{u_0}$, by comparison principle $\underline{u} \leq u$ and hence there exists $C > 0$ such that

$$\underline{u} \leq u \leq C, \qquad \text{for all} \;\; x \in \Omega, \;\; \text{and} \;\; t < T_{max}(u_0). \tag{4.9}$$

Also, because of the convexity of f and $f(0) = 0$,

$$\underline{u}_t - \Delta \underline{u} + h(|\nabla \underline{u}|) = f(\underline{u}) \geq f^{'}(0)\underline{u},$$

and by the maximum principle this implies

$$\underline{u}(x,t) \geq -C. \tag{4.10}$$

By (4.9) and (4.10) this proves that $\underline{u}$ is uniformly bounded in $(0, T_{max}(u_0))$ and we have

$$T_{max}(\underline{u_0}) \geq T_{max}(u_0).$$

Furthermore, by the strong maximum principle and Hopf maximum principle we have

$$u(x,\tau) > \underline{u}(x,\tau) \quad \text{in} \;\; \Omega \quad \text{and} \quad \frac{\partial u}{\partial \nu}(x,\tau) < \frac{\partial \underline{u}}{\partial \nu}(x,\tau) \quad \text{on} \;\; \partial\Omega, \;\; \tau \in (0, T_{max}(u_0))$$

Therefore, there exists $\alpha > 1$ such that $u(x,\tau) \geq \alpha \underline{u}(x,\tau)$ in Ω, $\tau \in (0, T_{max}(u_0))$. Now, in special cases $f(u) = u^p$ and $h(|\nabla u|) = |\nabla u|^q$ we have

$$\begin{aligned} f(\alpha \underline{u}) - h(|\nabla(\alpha \underline{u})|) &= (\alpha \underline{u})^p - |\nabla(\alpha \underline{u})|^q \\ &= \alpha^p \underline{u}^p - \alpha^q |\nabla \underline{u}|^q \\ &\geq \alpha^p \underline{u}^p - \alpha^p |\nabla \underline{u}|^q \\ &\geq \alpha(\underline{u}^p - |\nabla \underline{u}|^q), \end{aligned}$$

hence

$$(\alpha\underline{u})_t - \Delta(\alpha\underline{u}) - (\alpha\underline{u})^p + \mu|\nabla(\alpha\underline{u})|^q \leq \alpha(\underline{u}_t - \Delta\underline{u} - \underline{u}^p + \mu|\nabla\underline{u}|^q) = 0,$$

and by using comparison principle we obtain

$$u \geq \alpha\underline{u}, \qquad \tau \leq t < T_{max}(u_0).$$

□

Theorem 4.2.2. *Assume Ω is bounded, $p > 1$ and $p > q > \frac{2p}{p+1}$. Assume that (4.8) with $f(u) = u^p$, $h(|\nabla u|) = \mu|\nabla u|^q$, $\mu > 0$ and $p > q$ has a classical equilibrium v, with $v > 0$ in Ω. If $u_0 \in L^\infty(\Omega)$ satisfies $u_0 \geq v$, $u_0 \not\equiv v$, then $T_{max}(u_0) < \infty$.*

Proof. By Lemma 4.2.1 applied with $\underline{u_0} = v$, there exists $\alpha > 1$ and $\tau \in (0, T_{max}(u_0))$, such that

$$u(x,t) \geq \alpha v(x), \qquad t \in [\tau, T_{max}(u_0)). \tag{4.11}$$

Denote $z = z(t) = \int_\Omega u(t)v \text{ d}x$. Multiplying the equation in (4.8) with v, integrating by parts, using (3.12),(4.11) and by Hölder inequality, we obtain

$$\begin{aligned}
z' &= \int_\Omega u_t v \text{ d}x = \int_\Omega u\Delta v \text{ d}x + \int_\Omega u^p v \text{ d}x - \mu \int_\Omega |\nabla u|^q v \text{ d}x \\
&= \mu \int_\Omega u|\nabla v|^q \text{ d}x - \mu \int_\Omega v|\nabla u|^q \text{ d}x + \int_\Omega (u^p v - v^p u)\text{d}x \\
&= \mu \int_\Omega u|\nabla v|^q \text{ d}x - \mu \int_\Omega v|\nabla u|^q \text{d}x + \int_\Omega (1 - (v/u)^{p-1})u^p v \text{ d}x \\
&\geq \alpha\mu \int_\Omega v|\nabla v|^q \text{ d}x - \mu C \int_\Omega u^p v \text{ d}x + (1 - \alpha^{1-p}) \int_\Omega u^p v \text{ d}x \\
&\geq \alpha\mu \int_\Omega v|\nabla v|^q \text{ d}x - \mu C \Big(\int_\Omega v \text{ d}x\Big)^{1-p} z^p + (1 - \alpha^{1-p}) \Big(\int_\Omega v \text{ d}x\Big)^{1-p} z^p \\
&\geq C_1 + C_2 z^p \geq C_2 z^p,
\end{aligned}$$

for C small enough, $C_1 = C_1(v, \nabla v, \Omega)$, $C_2 = C_2(u, v, \Omega)$ and $t \in [\tau, T_{max}(u_0))$. It follows that u cannot exist globally. □

4.3 Liouville-type Theorems for Parabolic Equations with Gradient Terms

In this section we establish proofs of Liouville-type theorems for parabolic equations with gradient terms for $q > 2p/(p+1)$ in two cases.

4.3.1 The Case of Radial Solutions for $p < p_F$

In order to prove Liouville-type theorem for parabolic equation with a gradient term in the radial case, we need some preliminary observations concerning radial steady states. Let ψ_1 be the solution of the equation

$$\psi'' + \frac{n-1}{r}\psi' + \psi^p - |\psi'|^q = 0, \tag{4.12}$$

satisfying $\psi(0) = 1$, $\psi'(0) = 0$. It is known that the solution is defined on some interval and it changes sign due to $p \leq p_F$ (this follows from Theorem 2.4.1). We denote $r_1 > 0$ its first zero. By uniqueness for the initial value problem, it holds $\psi_1'(r_1) < 0$. We thus have

$$\psi_1(r) > 0, \qquad \text{in } [0, r_1) \quad \text{and} \quad \psi_1(r_1) = 0 > \psi_1'(r_1).$$

By scaling of ψ_1, we denote $\psi_\alpha(r) := \alpha\psi_1(\alpha^{\frac{p-q}{q}} r)$, which is the solution of (4.12) with $\psi(0) = \alpha$, $\psi'(0) = 0$, and with the first positive zero $r_\alpha = \alpha^{-\frac{p-q}{q}} r_1$.
As a result for the properties of ψ_1 we obtain the following lemma.

Lemma 4.3.1. *For any $m > 0$, we have*

$$\lim_{\alpha\to\infty} (\sup\{\psi_\alpha'(r) : \ r \in [0, r_\alpha] \text{ is such that } \psi_\alpha(r) \leq m\}) = -\infty.$$

We are ready now to prove Liouville-type result for the parabolic equation (4.1) by using arguments of intersection-comparison with (sign-changing) stationary solutions.

Theorem 4.3.2. *Let $1 < p < p_F$ and $q > \frac{2p}{p+1}$. Then (4.1) has no positive, radial, bounded classical solution.*

Proof. The proof is by contradiction. Assume that u is a positive, bounded classical solution of (4.1), $u(x,t) = U(r,t)$, where $r = |x|$.

By boundedness assumption and parabolic estimates, U and U_r are bounded on $[0,\infty)\times\mathbb{R}$. It follows from Lemma 4.3.1 that if α is sufficiently large, then $U(r,t)-\psi_\alpha$ has exactly one zero in $[0, r_\alpha]$ for any t and the zero is simple.

Next we claim that

$$z_{[0,r_\alpha]}(U(r,t) - \psi_\alpha) \geq 1, \quad t \leq 0, \ \alpha > 0, \tag{4.13}$$

where $z_{[0,\alpha]}(w)$ denotes the zero number of the function w in the interval $[0, r_\alpha]$. Indeed, if not, then $U(r,t_0) > \psi_\alpha$ in $[0, r_\alpha]$ for some t_0. By Theorem 4.2.2 we know that each solution of the Dirichlet problem

$$\left.\begin{aligned} \overline{u}_t - \Delta\overline{u} &= \overline{u}^p - \mu|\nabla\overline{u}|^q, && |x| < r_\alpha, \ t > 0, \\ \overline{u} &= 0, && |x| = r_\alpha, \ t > 0, \\ \overline{u}(x,t_0) &= \overline{U}_0(|x|), && |x| < r_\alpha \end{aligned}\right\}$$

blows up in finite time and $\overline{U}_0 > \psi_\alpha$ in $[0, r_\alpha)$. If we choose initial function $\overline{U}_0$ between ψ_α and $U(r,t_0)$ we obtain by comparison principle that $\overline{u}$ and u both blow up in finite time. This is contradiction to the global existence assumption on u, and this proves the claim.

We put

$$\alpha_0 := \inf\{\beta > 0 : \ z_{[0,r_\alpha]}(U(r,t) - \psi_\alpha) = 1 \ \text{ for all } \ t \leq 0 \ \text{ and } \ \alpha \geq \beta\}.$$

Considering large α, we get $\alpha_0 < \infty$. Also $\alpha_0 > 0$. Indeed, for small $\alpha > 0$ we have $\psi_\alpha(0) < U(0,t)$ for $t > 0$ small and for $t = 0$. By the properties of the zero number (see Theorem D.2.2), we can choose $t < 0$ small such that $\psi_\alpha(0) - U(r,t)$ has only simple zeros and then by (4.13), $z_{[0,r_\alpha]}(U(r,t) - \psi_\alpha) \geq 2$.

We conclude that there are sequences $\alpha_k \to \alpha_0$ and $t_k \leq 0$ such that

$$z_{[0,\alpha_k]}(U(r,t_k) - \psi_{\alpha_k}) \geq 2, \quad k = 1, 2, \dots.$$

and also by Theorem D.2.2, we obtain

$$z_{[0,\alpha_k]}(U(r,t_k+t)-\psi_{\alpha_k})\geq 2,\quad t\leq 0,\ k=1,2,.... \tag{4.14}$$

If we choose $t_k\to-\infty$, then by the boundedness assumption and parabolic estimates in Theorem C.2.1, passing to a subsequence, we can assume that

$$u(x,t_k+t)\to v(x,t),\quad x\in\mathbb{R}^n,\ t\in\mathbb{R},$$

which is converging in $C^{2,1}(\mathbb{R}^n\times\mathbb{R})$. Then there is $\delta>0$ such that for each fixed t,

$$U(r,t_k+t)-\psi_{\alpha_k}\to V(r,t)-\psi_{\alpha_0} \tag{4.15}$$

in $C^1[0,r_{\alpha_0}+\delta]$, since $V(|x|,t)=v(x,t)$. Then (4.14) and (4.15) give that for all $t\leq 0$, $V(r,t)-\psi_{\alpha_0}$ has at least two zeros or multiple zero in $[0,r_{\alpha_0})$.
By Theorem D.2.2, we can choose $t<0$ such that $V(r,t)-\psi_{\alpha_0}$ has only simple zeros (and hence at least two of them). Because $U(r,t_k+t)-\psi_{\alpha_0}\to V(r,t)-\psi_{\alpha_0}$ in $C^1[0,r_{\alpha_0}]$ if k is large, it has at least two simple zeros in $[0,r_{\alpha_0})$ as well. However, for $\alpha>\alpha_0$, $\alpha\to\alpha_0$, the function $u(r,t_k+t)-\psi_\alpha$ has at least two zeros in $[0,r_\alpha)$, this contradicts the definition of α_0.
Then we conclude that the assumption $u\not\equiv 0$ leads to a contradiction, which proves the theorem. □

4.3.2 The Case of Nonradial Solutions for $p<p_B$

The proof of Theorem 4.3.6 below is based on integral estimates for (local) positive solutions (cf. Proposition 4.3.5 below). Moreover, the proof of Proposition 4.3.5 is based on a key gradient estimate for local solutions of (4.1) (see (4.17) below). To establish this estimate, we prepare the following lemma, which provides a family of integral estimates relating any C^2-function with its gradient and its Laplacian. In the rest of this section we use the notation $\int\int=\int_{-T}^{T}\int_\Omega$ for simplicity.
For the proof of Lemma 4.3.4, we need the following lemma, which has been proved in [**47**].

Lemma 4.3.3. [**47**] *Let Ω be an arbitrary domain in $\mathbb{R}^n$, $0 \leq \phi \in D(\Omega)$, and $0 < u \in C^2(\Omega)$. Fix $q \in \mathbb{R}$ and denote*

$$I = \int_\Omega \phi u^{q-2}|\nabla u|^4 \mathrm{d}x, \quad J = \int_\Omega \phi u^{q-1}|\nabla u|^2(\Delta u)\mathrm{d}x, \quad K = \int_\Omega \phi u^q (\Delta u)^2 \mathrm{d}x.$$

Then, for any $k \in \mathbb{R}$ with $k \neq -1$, there holds

$$\alpha I + \beta J + \gamma K \leq \frac{1}{2}\int_\Omega u^q |\nabla u|^2(\Delta\phi)\mathrm{d}x + \int_\Omega u^q[\Delta u + (q-k)u^{-1}|\nabla u|^2](\nabla u \cdot \nabla\phi)\mathrm{d}x,$$

where

$$\alpha = -\frac{n-1}{n}k^2 + (q-1)k - \frac{q(q-1)}{2}, \quad \beta = \frac{n+2}{n}k - \frac{3q}{2}, \quad \gamma = -\frac{n-1}{n}.$$

Now, we turn to present Lemma 4.3.4 and its proof.

Lemma 4.3.4. *Let Ω be an arbitrary domain in $\mathbb{R}^n$, $T > 0$, and $0 \leq \phi \in \mathcal{D}(\Omega \times (-T,T))$. Let $0 < u \in C^{2,1}(\Omega \times (-T,T))$ be a solution of*

$$u_t - \Delta u = u^p - \mu|\nabla u|^q, \quad p > 1, \quad p > q > \frac{2p}{p+1} \quad \text{in } \Omega \times (-T,T). \tag{4.16}$$

Consider any $k \in \mathbb{R}$ with $k \neq -1$ and denote

$$I = \iint \phi u^{-2}|\nabla u|^4 \mathrm{d}x\mathrm{d}t, \qquad L = \iint \phi u^{2p}\mathrm{d}x\mathrm{d}t,$$

where the above and below integrals are over $\Omega \times (-T,T)$. Then, it holds

$$\begin{aligned}\alpha I + \delta L \leq C(n,p,k,\mu) &\iint \left\{\phi[(u_t)^2 + |u_t|u^{-1}|\nabla u|^2 + u^{p-1}|\nabla u|^2] + |\nabla u|^2|\Delta\phi|\right\}\mathrm{d}x\mathrm{d}t \\ + C(n,p,k,\mu) &\iint \left\{(u^p + |u_t| + u^{-1}|\nabla u|^2)|\nabla u.\nabla\phi| + u^{p+1}|\phi_t|\right\}\mathrm{d}x\mathrm{d}t, \end{aligned}\tag{4.17}$$

where

$$\alpha = -((n-1)k+n)\frac{k}{n}, \qquad \delta = -\frac{(n-1)(1+2\mu+\mu^2) + (n+2)(1+\mu)k/p}{n}. \tag{4.18}$$

Assume that $1 < p < p_B$. Then the constants α, δ defined in (4.18) satisfy $\alpha, \delta > 0$, where $k = k(n,p) \in \mathbb{R}$, $k \neq -1$.

Proof. i. We apply Lemma 4.3.3 with $q = 0$. Denoting

$$J = \iint \phi u^{-1}|\nabla u|^2 \Delta u \ \mathrm{d}x\mathrm{d}t, \qquad K = \iint \phi(\Delta u)^2 \ \mathrm{d}x\mathrm{d}t,$$

this gives us with using (3.12) and $\Delta u = u_t - u^p + \mu|\nabla u|^q$ that

$$\begin{aligned}
&-\left(\frac{n-1}{n}k+1\right)kI + \frac{n+2}{n}kJ - \frac{n-1}{n}K \\
&\leq \frac{1}{2}\iint |\nabla u|^2 \Delta\phi \ \mathrm{d}x\mathrm{d}t + \iint [\Delta u - ku^{-1}|\nabla u|^2]\nabla u \cdot \nabla\phi \ \mathrm{d}x\mathrm{d}t \\
&\leq \frac{1}{2}\iint |\nabla u|^2 \Delta\phi \ \mathrm{d}x\mathrm{d}t + \iint (u_t - u^p + \mu u^p - ku^{-1}|\nabla u|^2)\nabla u \cdot \nabla\phi \ \mathrm{d}x\mathrm{d}t.
\end{aligned} \tag{4.19}$$

Now, since $\Delta u = u_t - u^p + \mu|\nabla u|^q$, we obtain

$$\begin{aligned}
K = &\iint \phi(u_t)^2 \ \mathrm{d}x\mathrm{d}t + \iint \phi u^{2p} \ \mathrm{d}x\mathrm{d}t - 2\iint \phi u^p u_t \ \mathrm{d}x\mathrm{d}t \\
&+ 2\mu \iint \phi|\nabla u|^q u_t \ \mathrm{d}x\mathrm{d}t - 2\mu\iint \phi u^p|\nabla u|^q \ \mathrm{d}x\mathrm{d}t + \mu^2 \iint \phi|\nabla u|^{2q} \ \mathrm{d}x\mathrm{d}t.
\end{aligned}$$

Therefore,

$$\begin{aligned}
-K \geq &-\iint \phi(u_t)^2 \ \mathrm{d}x\mathrm{d}t - \iint \phi u^{2p} \ \mathrm{d}x\mathrm{d}t + 2\iint \phi u^p u_t \ \mathrm{d}x\mathrm{d}t \\
&- 2\mu C \iint \phi|\nabla u|^q u_t \ \mathrm{d}x\mathrm{d}t - 2\mu\iint \phi u^p|\nabla u|^q \ \mathrm{d}x\mathrm{d}t - \mu^2 \iint \phi|\nabla u|^{2q} \ \mathrm{d}x\mathrm{d}t
\end{aligned}$$

hence, by (3.12) and by integrating by parts in t we obtain,

$$-K \geq -\iint \phi(u_t)^2 \ \mathrm{d}x\mathrm{d}t - (1+2\mu+\mu^2)L - \left(\frac{2-2\mu C}{p+1}\right)\iint u^{p+1}\phi_t \ \mathrm{d}x\mathrm{d}t, \tag{4.20}$$

since $\Delta u = u_t - u^p + \mu|\nabla u|^q$, integrating by parts in x, and by (3.12), we have

$$\begin{aligned}
pJ = &-\iint \phi\nabla u \cdot \nabla(u^p) \ \mathrm{d}x\mathrm{d}t + p\iint \phi u_t u^{-1}|\nabla u|^2 \ \mathrm{d}x\mathrm{d}t \\
&\qquad + p\mu\iint \phi u^{-1}|\nabla u|^2|\nabla u|^q \ \mathrm{d}x\mathrm{d}t \\
= &\iint \phi(\Delta u)u^p \ \mathrm{d}x\mathrm{d}t + \iint (\nabla\phi \cdot \nabla u)u^p \ \mathrm{d}x\mathrm{d}t + p\iint \phi u_t u^{-1}|\nabla u|^2 \ \mathrm{d}x\mathrm{d}t \\
&\qquad + p\mu\iint \phi u^{-1}|\nabla u|^2|\nabla u|^q \ \mathrm{d}x\mathrm{d}t
\end{aligned}$$

$$\begin{aligned}
&= -\iint \phi u^{2p}\mathrm{d}x\mathrm{d}t + \iint \phi u^p u_t \,\mathrm{d}x\mathrm{d}t + \mu \iint \phi u^p |\nabla u|^q \mathrm{d}x\mathrm{d}t + \iint (\nabla\phi\cdot\nabla u) u^p \mathrm{d}x\mathrm{d}t \\
&\qquad + p\iint \phi u_t u^{-1}|\nabla u|^2 \,\mathrm{d}x\mathrm{d}t + p\mu \iint \phi u^{-1}|\nabla u|^2 |\nabla u|^q \,\mathrm{d}x\mathrm{d}t \\
&\geq -\iint \phi u^{2p}\,\mathrm{d}x\mathrm{d}t + \iint \phi u^p u_t \,\mathrm{d}x\mathrm{d}t - \mu \iint \phi u^{2p}\,\mathrm{d}x\mathrm{d}t + \iint (\nabla\phi\cdot\nabla u) u^p \,\mathrm{d}x\mathrm{d}t \\
&\qquad + p\iint \phi u_t u^{-1}|\nabla u|^2 \,\mathrm{d}x\mathrm{d}t - p\mu \iint \phi u^{p-1}|\nabla u|^2 \,\mathrm{d}x\mathrm{d}t,
\end{aligned}$$

by integrating by parts in t we obtain

$$\begin{aligned}
pJ \geq -(1+\mu)L - \left(\frac{1}{p+1}\right) \iint u^{p+1}\phi_t \,\mathrm{d}x\mathrm{d}t + \iint (\nabla\phi\cdot\nabla u)u^p \,\mathrm{d}x\mathrm{d}t \\
+p\iint \phi u_t u^{-1}|\nabla u|^2 \,\mathrm{d}x\mathrm{d}t - p\mu \iint \phi u^{p-1}|\nabla u|^2 \,\mathrm{d}x\mathrm{d}t.
\end{aligned} \tag{4.21}$$

Substituting (4.20) and (4.21) in (4.19),

$$\begin{aligned}
&-\left(\frac{n-1}{n}k+1\right)kI + \left\{\left(\frac{n+2}{n}\right)k\big[-\frac{1}{p}(1+\mu)\big] - \left(\frac{n-1}{n}\right)(1+2\mu+\mu^2)\right\}L \\
&\leq C(n,p,k,\mu)\iint \{\phi[(u_t)^2 + |u_t|u^{-1}|\nabla u|^2 + u^{p-1}|\nabla u|^2] + |\nabla u|^2|\Delta\phi|\}\mathrm{d}x\mathrm{d}t \\
&\qquad + C(n,p,k,\mu)\iint \big\{(u^p + |u_t| + u^{-1}|\nabla u|^2)|\nabla u\cdot\nabla\phi| + u^{p+1}|\phi_t|\big\}\mathrm{d}x\mathrm{d}t,
\end{aligned}$$

where

$$\begin{aligned}
\alpha &= -\left[\frac{n-1}{n}k+1\right]k = -[(n-1)k+n]\frac{k}{n}, \\
\delta &= \left\{\left(\frac{n+2}{n}\right)k\left(-\frac{1+\mu}{p}\right) - \left(\frac{n-1}{n}\right)(1+2\mu+\mu^2))\right\} \\
&= -\frac{(n-1)(1+2\mu+\mu^2) + (n+2)(1+\mu)k/p}{n}
\end{aligned}$$

In order to be $\delta > 0$, we can say that

$$-\frac{(n-1)+(n+2)k/p}{n} > -\frac{(n-1)(1+2\mu+\mu^2)+(n+2)(1+\mu)k/p}{n} = \delta > 0$$

ii. Therefore, for $k < 0$, the condition $\alpha, \delta > 0$ is equivalent to

$$\frac{(n-1)p}{n+2} < -k < \frac{n}{n-1}.$$

This choice of $k < 0$ is possible if $p < p_B$.

□

Proposition 4.3.5. *Let $1 < p < p_B$ and let B_1 be the unit ball in $\mathbb{R}^n$. There exist $r = r(n,p) > (n+2)(p-1)/2$ such that if $0 < u \in C^{2,1}(B_1 \times (-1,1))$ is a solution of*

$$u_t - \Delta u = u^p - \mu|\nabla u|^q, \quad p > 1, \;\; p > q > \frac{2p}{p+1}, \quad |x| < 1, \quad -1 < t < 1,$$

then

$$\int_{-1/2}^{1/2} \int_{|x|<1/2} u^r \mathrm{d}x\mathrm{d}t \le C(n,p). \tag{4.22}$$

Proof. Taking k as in Lemma 4.3.4(ii), we will estimate the terms on the RHS of (4.17). Firstly, we shall prepare a suitable test function, and we take $\xi \in D(B_1 \times (-1,1))$ such that $\xi = 1$ in $B_{1/2} \times (-1/2, 1/2)$ and $0 \le \xi \le 1$.
By taking $\phi = \xi^{\frac{4p}{p-1}}$, we have

$$|\nabla\phi| \le C\phi^{\frac{3p+1}{4p}}, \qquad |\Delta\phi| \le C\phi^{\frac{p+1}{2p}}, \qquad |\phi_t| \le C\phi^{\frac{3p+1}{4p}} \le \phi^{\frac{p+1}{2p}}. \tag{4.23}$$

Secondly, we notice that

$$\iint |\nabla u|^2(|\Delta\phi| + \phi^{-1}|\nabla\phi|^2 + |\phi_t|)\mathrm{d}x\mathrm{d}t \le \eta(I+L) + C(\eta), \qquad \eta > 0. \tag{4.24}$$

Indeed, this follows from Young's inequality and (4.23), by writing

$$\begin{aligned}
|\nabla u|^2(|\Delta\phi| + \phi^{-1}|\nabla\phi|^2 + |\phi_t|) &= uu^{-1}\phi^{1/2}\phi^{-1/2}|\nabla u|^2(|\Delta\phi| + \phi^{-1}|\nabla\phi|^2 + |\phi_t|) \\
&\le \eta\phi u^{-2}|\nabla u|^4 + C(\eta)\phi^{-1}u^2(|\Delta\phi| + \phi^{-1}|\nabla\phi|^2 + |\phi_t|)^2 \\
&\le \eta\phi u^{-2}|\nabla u|^4 + C(\eta)\phi^{-1}u^2(3\phi^{\frac{p+1}{2p}})^2 \\
&= \eta\phi u^{-2}|\nabla u|^4 + C(\eta)\phi^{\frac{1}{p}}u^2 \\
&\le \eta\phi u^{-2}|\nabla u|^4 + \eta\phi u^{2p} + C(\eta).
\end{aligned}$$

Now, we fix $\epsilon > 0$. Using Young's inequality, (4.23) and (4.24), we estimate the RHS of (4.17) as follows

$$\begin{aligned}
&\iint \left\{\phi[(u_t)^2 + |u_t|u^{-1}|\nabla u|^2 + u^{p-1}|\nabla u|^2] + |\nabla u|^2|\Delta\phi|\right\}\mathrm{d}x\mathrm{d}t \\
&\qquad + \iint \left\{(u^p + |u_t| + u^{-1}|\nabla u|^2)|\nabla u \cdot \nabla\phi| + u^{p+1}|\phi_t|\right\}\mathrm{d}x\mathrm{d}t
\end{aligned}$$

$$
\begin{aligned}
&\leq \epsilon \iint \phi[u^{2p} + u^{-2}|\nabla u|^4]\mathrm{d}x\mathrm{d}t \\
&\qquad + C(\epsilon) \iint [\phi(u_t)^2 + |\nabla u|^2(\phi^{-1}|\nabla \phi|^2 + |\Delta \phi|) + (\phi^{-(p+1)}|\phi_t|^{2p})^{\frac{1}{p-1}}]\mathrm{d}x\mathrm{d}t \\
&\leq 2\epsilon(I+L) + C(\epsilon)\{1 + \iint \phi(u_t)^2 \, \mathrm{d}x\mathrm{d}t\}. \qquad (4.25)
\end{aligned}
$$

For treating the last term in the above inequality, we will multiply the equation (4.16) by ϕu_t and integrate by parts in x, we use (3.12), and integrate by parts in t, we have

$$
\begin{aligned}
\iint \phi(u_t)^2 \, \mathrm{d}x\mathrm{d}t &= \iint \phi u_t \Delta u \, \mathrm{d}x\mathrm{d}t + \iint \phi u^p u_t \, \mathrm{d}x\mathrm{d}t - \mu \iint \phi |\nabla u|^q u_t \, \mathrm{d}x\mathrm{d}t \\
&\leq \iint \phi u_t \Delta u \, \mathrm{d}x\mathrm{d}t + (1+|\mu|) \iint \phi u^p u_t \, \mathrm{d}x\mathrm{d}t \\
&= \iint \big\{\phi \partial_t \big[(1+|\mu|)\frac{u^{p+1}}{p+1} - \frac{|\nabla u|^2}{2}\big] - (\nabla\phi \cdot \nabla u)u_t\big\}\mathrm{d}x\mathrm{d}t.
\end{aligned}
$$

By integrating by parts in t, and then by using Young's inequality, we obtain

$$
\begin{aligned}
\iint \phi(u_t)^2 \mathrm{d}x\mathrm{d}t &= \iint \big\{\big[\frac{|\nabla u|^2}{2} - (1+|\mu|)\frac{u^{p+1}}{p+1}\big]\phi_t - (\nabla\phi \cdot \nabla u)u_t\big\}\mathrm{d}x\mathrm{d}t \\
&\leq \frac{1}{2}\iint |\nabla u|^2(|\phi_t| + |\nabla\phi|^2\phi^{-1})\mathrm{d}x\mathrm{d}t + \frac{1}{2}\iint \phi(u_t)^2\mathrm{d}x\mathrm{d}t \\
&\qquad + \frac{(1+|\mu|)}{p+1}\iint u^{p+1}|\phi_t|\mathrm{d}x\mathrm{d}t.
\end{aligned}
$$

Thus by (4.24) and (4.23), and by Young's inequality for $\eta > 0$, we have

$$
\begin{aligned}
\iint \phi(u_t)^2 \mathrm{d}x\mathrm{d}t &\leq \iint |\nabla u|^2(|\phi_t| + |\nabla\phi|^2\phi^{-1})\mathrm{d}x\mathrm{d}t + \frac{2(1+|\mu|)}{p+1}\iint u^{p+1}|\phi_t|\mathrm{d}x\mathrm{d}t \\
&\leq \eta(I+L) + C(\eta) + \eta\iint \phi u^{2p}\mathrm{d}x\mathrm{d}t + C(\eta)\iint \phi^{-\frac{p+1}{p-1}}|\phi_t|^{\frac{2p}{p-1}}\mathrm{d}x\mathrm{d}t \\
&\leq 2\eta(I+L) + C(\eta). \qquad (4.26)
\end{aligned}
$$

Combining (4.26) with $\eta = \epsilon(2C(\epsilon))^{-1}$, (4.25) and (4.17), we will have

$$
\alpha I + \delta L \leq C(n,p)\epsilon(I+L) + C(\epsilon).
$$

Since $\alpha, \delta > 0$, and by choosing ϵ sufficiently small, we conclude that $I, L \leq C$. $\square$

Now, we state Liouville-type theorems for parabolic problems with a gradient term for $q > 2p/(p+1)$. The proof of Theorem 4.3.6 is a direct consequence of the space-time integral estimates (4.22) for (local) solutions of (4.1). It is based on the simple homogeneity argument.

Theorem 4.3.6. *Let $1 < p < p_B$, then equation (4.16) has no positive classical solution.*

Proof. Let $R > 0$. Let u be a solution of (4.16). Then for each $R > 0$, $v(x,t) = R^{2/(p-1)}u(Rx, R^2t)$ solves (4.16) in $B_1 \times (-1,1)$. It follows from Proposition 4.3.5 that

$$\begin{aligned}\int_{-R^2/2}^{R^2/2}\int_{|y|<R/2} u^r(y,s)\mathrm{d}y\mathrm{d}s &= R^{n+2}\int_{-1/2}^{1/2}\int_{|x|<1/2} u^r(Rx,R^2t)\mathrm{d}x\mathrm{d}t \\ = R^{n+2-2r/(p-1)}\int_{-1/2}^{1/2}\int_{|x|<1/2} v^r(x,t)\mathrm{d}x\mathrm{d}t &\le C(n,p)R^{n+2-2r/(p-1)}\end{aligned}$$

Since $r > (n+2)(p-1)/2$, and by letting $R \to \infty$, we conclude that $\int_{-\infty}^{\infty}\int_{\mathbb{R}^n} u^r \mathrm{d}y\mathrm{d}s = 0$, hence $u \equiv 0$. $\square$

Remark 4.3.7. *It is clear that the estimate (3.21) implies nonexistence of positive solution of (4.1). On the other hand, if Theorem 4.3.6 were known for all $p > 1$ and $p > q > \frac{2p}{p+1}$, then this would imply Theorem 3.3.5 as well.*
Moreover, Theorem 3.3.5 guarantees that Theorem 4.3.6 remains true for nontrivial nonnegative radial classical solutions bounded or not.

4.4 A priori Bounds and Blow-up Rates

In this section we will consider the a priori bounds and blow up rates in two cases:

i. for $q = 2p/(p+1)$

ii. for $q > 2p/(p+1)$

4.4.1 Case i: $q = 2p/(p+1)$

In this subsection we use a general method which unifies and improves many results concerning *universal* (independent of the solution itself and even possibly of the domain) a priori estimates of global solutions, blow up rates of non-global solutions, initial blow-up rates of local solutions, decay rates of global solutions and spatial singularity estimates for local solutions of the perturbed problem (4.5). This method is based on a doubling lemma, a rescaling argument, and the parabolic Liouville-type theorems. The solution that we consider are defined on an arbitrary spatial domain, without any prescribed initial conditions, but they may or may not satisfy boundary conditions.

In order to show that the universal bounds of the form (4.7) are also satisfied for the problem (4.5) when $q = 2p/(p+1)$, we need to recall the doubling lemma from [**44**], and Liouville-type theorems from [**43**] and [**44**].

Lemma 4.4.1. [**44**] *Let (X, d) be a complete metric space and let $\phi \neq D \subset \Sigma \subset X$, with Σ closed. Set $\Gamma = \Sigma \backslash D$. Finally let $M : D \to (0, \infty)$ be bounded on compact subsets of D and fix a real $k > 0$. If there exists $y \in D$ such that*

$$M(y)dist(y, \Gamma) > 2k$$

then there exists $x \in D$ such that

$$M(x)dist(x, \Gamma) > 2k, \qquad M(x) \geq M(y),$$

and

$$M(z) \leq 2M(x) \quad \text{for all} \ \ z \in D \cap \overline{B}_X(x, kM^{-1}(x)).$$

Theorem 4.4.2. [**43**] *Let $1 < p < p_S$. Then the equation*

$$u_t - \Delta u = u^p, \qquad x \in \mathbb{R}^n, \quad t \in \mathbb{R},$$

has no positive, radial, bounded classical solution.

Theorem 4.4.3. [**44**] *Let $p > 1$. Assume $n \leq 2$, or $p < (n-1)(n+1)/(n-2)^2$ and $n \geq 3$. Then the problem*

$$\left.\begin{aligned} u_t - \Delta u = u^p, \qquad & x \in \mathbb{R}^n_+, \quad t \in \mathbb{R}, \\ u = 0, \qquad & x \in \partial\mathbb{R}^n_+, \quad t \in \mathbb{R} \end{aligned}\right\} \tag{4.27}$$

has no positive bounded classical solution.

The following result shows that universal bounds of the problem (4.5) when $q < 2p/(p+1)$ are the same as that of the problem (4.5) when $q = 2p/(p+1)$.

Theorem 4.4.4. *Let $p > 1$ and $T > 0$. Assume that either*

$$p < p_B, \quad \text{or} \quad p < p_S, \qquad \Omega = \mathbb{R}^n \text{ or } B_R, \quad u = u(|x|, t), \quad g = g(u, |\xi|). \tag{4.28}$$

Assume in addition that the function $g : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}$ satisfies the growth assumption

$$\begin{gathered} |g(u,\xi)| \leq C_0(1 + |u|^{p_1} + |\xi|^q) \\ \text{for some} \quad 1 \leq p_1 \leq p \quad \text{and} \quad q = 2p/(p+1). \end{gathered} \tag{4.29}$$

Then for any non negative classical solution of (4.5), there holds

$$\begin{gathered} u(x,t) + |\nabla u(x,t)|^{\frac{2}{p+1}} \leq C(1 + t^{\frac{-1}{p-1}} + (T-t)^{\frac{-1}{p-1}}), \qquad x \in \Omega, \quad 0 < t < T, \\ \text{with} \quad C = C(p, p_1, q, C_0, \Omega) > 0. \end{gathered}$$

Proof. Assume the contrary. Then there exist sequences $T_k \in (0, \infty)$, $u_k, y_k \in \Omega$, $s_k \in (0, T_k)$ such that u_k solves (4.5), and the function

$$M_k = u_k^{\frac{p-1}{2}} + |\nabla u_k|^{\frac{p-1}{p+1}} \tag{4.30}$$

satisfies

$$M_k(y_k, s_k) > 2k(1 + d_k^{-1}(s_k)), \tag{4.31}$$

where $\quad d_k(t) = \min(t, T_k - t)^{\frac{1}{2}}$. We will use Lemma 4.4.1 with $X = \mathbb{R}^{n+1}$ equipped with the parabolic distance

$$d_p[(x,t),(y,s)] = |x-y| + |t-s|^{\frac{1}{2}},$$

$$\Sigma = \Sigma_k = \bar{\Omega} \times [0, T_k], \quad D = D_k = \bar{\Omega} \times (0, T_k) \ \text{ and } \ \Gamma = \Gamma_k = \bar{\Omega} \times \{0, T_k\}$$

$$d_k(t) = dist_p((x,t), \Gamma_k), \quad (x,t) \in \Sigma_k.$$

By Lemma 4.4.1, there exist $x_k \in \Omega$, $t_k \in (0, T_k)$ such that

$$M_k(x_k, t_k) > 2k d_k^{-1}(t_k), \tag{4.32}$$

$$M_k(x_k, t_k) > M_k(y_k, s_k) > 2k$$

and

$$M_k(x,t) \leq 2M_k(x_k, t_k), \qquad (x,t) \in D_k \cap \tilde{B}_k, \tag{4.33}$$

where

$$\tilde{B}_k = \{(x,t) \in \mathbb{R}^{n+1} : |x - x_k| + |t - t_k|^{\frac{1}{2}} \leq k\lambda_k\}$$

and

$$\lambda_k = M_k^{-1}(x_k . t_k) \to 0 \tag{4.34}$$

and by (4.32) for all $(x,t) \in \tilde{B}_k$, we have

$$|t - t_k| \leq k^2 \lambda_k^2 < d_k^2(t_k) = \min\{t_k, T_k - t_k\}, \qquad t \in (0, T_k).$$

It follows that

$$\Big(\Omega \cap \big\{|x - x_k| < \frac{k\lambda_k}{2}\big\}\Big) \times \Big(t_k - \frac{k^2\lambda_k^2}{4}, t_k + \frac{k^2\lambda_k^2}{4}\Big) \subset D_k \cap \tilde{B}_k.$$

Now we rescale u_k by setting

$$v_k(y,s) = \lambda_k^{\frac{2}{p-1}} u_k(x_k + \lambda_k y, t_k + \lambda_k^2 s), \qquad (y,s) \in \tilde{D}_k, \tag{4.35}$$

where

$$\tilde{D}_k := \Big(\lambda_k^{-1}(\Omega - x_k) \cap \big\{|y| < \frac{k}{2}\big\}\Big) \times \Big(-\frac{k^2}{4}, \frac{k^2}{4}\Big).$$

The function v_k solves

$$\left.\begin{aligned} \partial_s v_k - \Delta_y v_k &= v_k^p + g_k, && (y,s) \in \tilde{D}_k, \\ v_k &= 0, && y \in \lambda_k^{-1}(\partial\Omega - x_k) \cap |y| < \frac{k}{2}, |s| < \frac{k^2}{4}, \end{aligned}\right\} \tag{4.36}$$

with

$$g_k(y,s) = \lambda_k^{\frac{2p}{p-1}} g\Big(\lambda_k^{\frac{-2}{p-1}} v_k(y,s), \lambda_k^{-\frac{p+1}{p-1}} \nabla v_k(y,s)\Big).$$

Moreover we have

$$v_k^{\frac{p-1}{2}}(0) + |\nabla v_k|^{\frac{p-1}{p+1}}(0) = \lambda_k u_k^{\frac{p-1}{2}}(x_k, \lambda_k) + \lambda_k^p |\nabla u_k(x_k, t_k)|^{\frac{p-1}{p+1}} = 1, \tag{4.37}$$

and (4.33) implies

$$v_k^{\frac{p-1}{2}}(y,s) + |\nabla v_k|^{\frac{p-1}{p+1}}(y,s) \le 2, \qquad (y,s) \in \tilde{D}_k. \tag{4.38}$$

The growth assumption (4.29) implies

$$|g_k| \le C\lambda_k^m, \qquad m = \min\Big\{\frac{2(p-p_1)}{p-1}, \frac{2p - q(p+1)}{p-1}\Big\} = 0.$$

Let $\rho_k = dist(x_k, \partial\Omega)$, then after passing to a subsequence either

$$\rho_k/\lambda_k \to \infty \tag{4.39}$$

or

$$\rho_k/\lambda_k \to C \ge 0. \tag{4.40}$$

In case (4.39):

By using (4.36), (4.38), (4.34), L^p estimates and the embedding (C.9), we deduce that some subsequence of v_k converges in $C^{1+\sigma,\frac{\sigma}{2}}(\mathbb{R}^n \times \mathbb{R})$, $\sigma \in (0,1)$ to a bounded classical solution $0 \le v \in W^{2,1;r}(\mathbb{R}^n \times \mathbb{R})$, $1 < r < \infty$, which satisfies $v_s - \Delta v \le v^p + C$. Therefore we get $v^{\frac{p-1}{2}}(0) + |\nabla v|^{\frac{p-1}{p+1}}(0) = 1$, so that v is non-trivial, moreover, v and ∇v are bounded. Since $v_s - \Delta v - v^p - C \le 0 = u_t - \Delta u - u^p$, then by comparison principle we have $v \le u$, since $u \ge 0$ is the solution of (4.4) with $u(0,0) \ge 1$, also v

satisfies $v_s - \Delta v \geq v^p - C$. Furthermore, $v_s - \Delta v - v^p + C \geq 0 = u_t - \Delta u - u^p$, and also by strong comparison principle (Proposition D.1.4) we obtain that $v \geq u$. We conclude that $0 \leq v = u$.

As a consequence of strong maximum principle (Proposition D.1.4), we have either $u > 0$ in $\mathbb{R}^{n+1}$, or

$$u = 0 \text{ in } \mathbb{R}^n \times (-\infty, t_0] \quad \text{and} \quad u > 0 \text{ in } Q := \mathbb{R}^n \times (t_0, \infty), \tag{4.41}$$

for some $t_0 < 0$. Sine $u \leq C$ due to (4.38), then in the latter case we have $u_t - \Delta u \leq C^{p-1}u$ in Q and by using maximum principle in Proposition D.1.1, we have that $u = 0$ in Q, which is a contradiction. Then $u > 0$ contradicts Theorem 4.4.2.

In the case (4.40):

Let $\tilde{x_j} \in \partial\Omega$ be such that $d_j = |x_j - \tilde{x_j}|$ and let R_j be the orthogonal transformation in $\mathbb{R}^n$ that maps $-e_1 = (-1, 0, ..., 0)$ onto the outer normal vector to $\partial\Omega$ at $\tilde{x_j}$. Now we define

$$v_k(y,s) = \lambda_k^{\frac{2}{p-1}} u(\lambda_k R_j y + x_k, \lambda_k^2 s + t_k)$$

$$\text{for} \quad (y,s) \in \tilde{\bar{D}}_k \times \Big(-\frac{k^2}{4}, \frac{k^2}{4}\Big),$$

$$\text{where} \quad \tilde{D_k} = \{y \in \mathbb{R}^n : \lambda_k R_j y + x_k \in \Omega\}.$$

Then v_k is solution of

$$\begin{aligned} \partial_s v_k - \Delta_k v_k = v_k^p + g_k \quad & \textit{in} \ \ \tilde{D_k} \times \Big(-\frac{k^2}{4}, \frac{k^2}{4}\Big) \\ v_k = 0 \qquad & \textit{on} \ \ \partial\tilde{D_k} \times \Big(-\frac{k^2}{4}, \frac{k^2}{4}\Big) \end{aligned}$$

with

$$g_k(y,s) = \lambda_k^{\frac{2p}{p-1}} g\Big(\lambda_k^{\frac{-2}{p-1}} v_k(y,s), \lambda_k^{-\frac{p+1}{p-1}} \nabla v_k(y,s)\Big).$$

Clearly $\tilde{D_k}$ approaches (locally) the half space $H_c = \{y_1 > -c\}$ as $\lambda_k \to 0$.

From (4.36), (4.38), (4.34), interior estimates and embedding (C.9) we obtain subsequence v_k which converges in $C^\alpha(\bar{H}_c)$, $0 < \alpha < 1$ to a solution of

$$\partial_s v - \Delta_y v = l v^p + g, \quad y \in H_c, \quad s \in \mathbb{R}$$

$$v = 0 \qquad y \in \partial H_c, \quad s \in \mathbb{R}$$

with $v(0,0) = \lambda_k^{\frac{2}{p-1}} u(\lambda_k \tilde{x_j} + x_k, t_k) = 1$. Similarly as in the last case we obtain $v > 0$ which contradicts Theorem 4.4.3 $\square$

As an interesting consequence of Theorem 4.4.4 in the case of $\mathbb{R}^n$, we obtain the universal decay of all nonnegative global solutions of (4.5) in $\mathbb{R}^n \times (0, \infty)$.

Corollary 4.4.5. *Let $p > 1$ and u be a global (non-negative) solution of (4.5) on $\mathbb{R}^n \times (0, \infty)$. Assume (4.29) and*

$$p < p_B, \quad \text{or} \quad p < p_S, \quad u = u(|x|, t), \quad g = g(u, |\xi|). \tag{4.42}$$

Then it holds

$$u(x,t) \leq C(n,p) t^{-\frac{1}{p-1}}, \qquad x \in \mathbb{R}^n, \qquad t > 0.$$

4.4.2 Case ii: $q > 2p/(p+1)$

Deriving an universal a priori bound for global solution and the blow up rate estimate based on deriving some basic estimates for positive solutions of (4.6) for $q > 2p/(p+1)$. These estimates can be shown in the next lemma.

Lemma 4.4.6. *Assume Ω bounded and convex, $p > 1$, $p > q > 2p/(p+1)$, $0 \leq \phi_1 \in D(\Omega \times (-T, T))$, and $0 < T < \infty$. Let u be a nonnegative classical solution of (4.6) on $(0, T)$. Then for all $t \in (0, T/2]$, there holds*

$$\int_\Omega u(x,t)\phi_1 \mathrm{d}x \leq C(p, \Omega)(1 + T^{-1/(p-1)}), \tag{4.43}$$

and

$$\int_0^t \int_\Omega (u^p(x,s) - \mu |\nabla u(x,s)|^q)\phi_1 \mathrm{d}x\mathrm{d}s \leq C(p,\Omega)(1+t)(1 + T^{-1/(p-1)}). \tag{4.44}$$

Proof. Denote $y = y(t) := \int_\Omega u(t)\phi_1 \mathrm{d}x$, multiplying the equation (4.6) by ϕ_1, integrating by parts, using $\Delta\phi_1 = -\phi_1$, and (3.12). We obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u(t)\phi_1 \mathrm{d}x - \int_\Omega \Delta u(t)\phi_1 \mathrm{d}x = \int_\Omega u^p(t)\phi_1 \mathrm{d}x - \mu \int_\Omega |\nabla u(t)|^q \phi_1 \mathrm{d}x,$$

hence

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u(t)\phi_1\mathrm{d}x &= \int_\Omega u^p(t)\phi_1\mathrm{d}x - \mu\int_\Omega |\nabla u(t)|^q\phi_1\mathrm{d}x - \int_\Omega u\phi_1\mathrm{d}x \\ &\geq \int_\Omega u^p(t)\phi_1\mathrm{d}x - C\int_\Omega u^p(t)\phi_1\mathrm{d}x - \int_\Omega u(t)\phi_1\mathrm{d}x. \qquad (4.45)\end{aligned}$$

By Jensen's inequality, we obtain that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u(t)\phi_1\mathrm{d}x \geq C_1\Big(\int_\Omega u(t)\phi_1\mathrm{d}x\Big)^p - \int_\Omega u(t)\phi_1\mathrm{d}x.$$

Since u exists on $(0,T)$, we deduce that

$$\int_\Omega u(t)\phi_1\mathrm{d}x \leq C_2(p,\Omega)(1+(T-t)^{-1/(p-1)}), \quad 0<t<T.$$

Now, integrating (4.45) in time over (τ,t), $0<\tau<t<T/2$ and using (4.43), we obtain

$$\begin{aligned}\int_\tau^t\int_\Omega\Big(u^p(s)-\mu|\nabla u(s)|^q\Big)\phi_1\mathrm{d}x\mathrm{d}s &= \int_\tau^t\int_\Omega u(s)\phi_1\mathrm{d}x\mathrm{d}s + \int_\Omega u(t)\phi_1\mathrm{d}x - \int_\Omega u(\tau)\phi_1\mathrm{d}x \\ &\leq C(p,\Omega)(1+t)(1+T^{-1/(p-1)}),\end{aligned}$$

by letting $\tau\to 0$, we will have (4.44). □

In Lemma 4.4.8 below, we will show a further estimate for positive solution of (4.6) when $q>2p/(p+1)$, whose proof uses the special test-function constructed in the following lemma (see [**8**] and [**48**]) by considering a singular elliptic problem.

Lemma 4.4.7. [**8, 48**] *Assume Ω bounded and $0<\alpha<1$. Then the problem*

$$\left.\begin{aligned}-\Delta\xi &= \phi_1^{-\alpha}, \qquad x\in\Omega, \\ \xi &= 0, \qquad x\in\partial\Omega,\end{aligned}\right\} \qquad (4.46)$$

admits a unique classical solution $\xi\in C(\bar\Omega)\cap C^2(\Omega)$. Moreover, we have $\phi_1^{-\alpha}\in L^1(\Omega)$, $\xi\in H_0^1(\Omega)$, and

$$\xi(x)\leq C(\Omega,\alpha)\delta(x), \qquad x\in\Omega.$$

Lemma 4.4.8. *Assume Ω bounded and convex, $p > 1$, $p > q > 2p/(p+1)$, $0 < T < \infty$ and $\varepsilon \in (0, (p+1)/2]$. Let u be a nonnegative classical solution of (4.6) on $(0, \tau)$. Then for all $t \in (0, T/2]$, there holds*

$$\int_0^t \int_\Omega u^{\frac{p+1}{2}-\varepsilon} \mathrm{d}x\mathrm{d}s \leq C(p, \Omega, \varepsilon)(1+t)(1+T^{-1/(p-1)}).$$

Proof. For given $0 < \alpha < 1$, Lemma 4.4.7 guarantees the existence of a function $\xi \in C(\bar{\Omega}) \cap C^2(\Omega) \cap H_0^1(\Omega)$ such that $-\Delta\xi = \phi_1^{-\alpha}$ in Ω. Moreover, ξ satisfies

$$\xi(x) \leq C(\Omega, \alpha)\delta(x), \qquad x \in \Omega. \tag{4.47}$$

We choose $\alpha = \frac{r'}{r}$, where r is defined by

$$\frac{1}{r} = \frac{1}{2} - \frac{\varepsilon}{p-1}, \qquad \frac{1}{r'} = 1 - \frac{1}{r} = \frac{1}{2} + \frac{\varepsilon}{p-1}.$$

Taking ξ as a test function in (4.6) and integrating over (τ, t), we obtain

$$\int_\tau^t \int_\Omega u\phi_1^{-\alpha} \mathrm{d}x\mathrm{d}s = \int_\tau^t \int_\Omega \left[u^p - \mu|\nabla u|^q\right]\xi \mathrm{d}x\mathrm{d}s + \int_\Omega u(\tau)\xi \mathrm{d}x - \int_\Omega u(t)\xi \mathrm{d}x.$$

Due to (4.43), (4.44) and (4.47), we have

$$\begin{aligned}\int_\tau^t \int_\Omega u\phi_1^{-\alpha} \mathrm{d}x\mathrm{d}s &\leq C\int_\tau^t \int_\Omega \left[u^p - \mu|\nabla u|^q\right]\delta(x) \mathrm{d}x\mathrm{d}s + C\int_\Omega u(\tau)\delta(x) \mathrm{d}x - C\int_\Omega u(t)\delta(x) \mathrm{d}x. \\ &\leq C\int_\tau^t \int_\Omega \left[u^p - \mu|\nabla u|^q\right]\phi_1 \mathrm{d}x\mathrm{d}s + C\int_\Omega u(\tau)\phi_1 \mathrm{d}x - c\int_\Omega u(t)\phi_1 \mathrm{d}x,\end{aligned}$$

hence

$$\int_\tau^t \int_\Omega u\phi_1^{-\alpha} \mathrm{d}x\mathrm{d}s \leq C(p, \Omega, \varepsilon)(1+t)(1+T^{-1/(p-1)}).$$

By using Hölder's inequality, the last estimate and (4.44) imply

$$\begin{aligned}\int_0^t \int_\Omega u^{\frac{p+1}{2}-\varepsilon} &= \int_0^t \int_\Omega \left(u^{p/r}\phi_1^{1/r}\right)\left(u^{1/r'}\phi_1^{-1/r}\right) \mathrm{d}x\mathrm{d}t \\ &\leq \Big(\int_0^t \int_\Omega u^p\phi_1 \mathrm{d}x\mathrm{d}t\Big)^{1/r}\Big(\int_0^t \int_\Omega u\phi_1^{-\alpha} \mathrm{d}x\mathrm{d}t\Big)^{1/r'} \\ &\leq C(p, \Omega, \varepsilon)(1+t)(1+T^{-1/(p-1)}).\end{aligned}$$

□

Further theorems can be used in the proof of Theorem 4.4.12 below; see Theorems 4.4.9, 4.4.10 and 4.4.11 below, which are from [**47**], [**25**] and [**51**], respectively.

Theorem 4.4.9. [**47**] *Let $p > 1$, $u_0 \in L^q(\Omega)$, $1 \leq q < \infty$, $q > n(p-1)/2$. Then there exists $T = T(\|u_0\|_q) > 0$ such that problem (3.4) possesses a unique classical L^q-solution in $[0, T)$ and the following smoothing estimate is true*

$$\|u(t)\|_r \leq C \|u_0\|_q t^{-\alpha_r}, \qquad \alpha_r := \frac{n}{2}\Big(\frac{1}{q} - \frac{1}{r}\Big), \tag{4.48}$$

for all $t \in (0, T)$ and $r \in [q, \infty]$, with $C = C(n, p, q) > 0$. In addition, $u \geq 0$ provided $u_0 \geq 0$.

Theorem 4.4.10. [**25**] *Let $1 < p \leq p_S$. Then the problem*

$$\left.\begin{aligned} -\Delta u &= u^p, && x \in \mathbb{R}^n_+, \\ u &= 0, && x \in \partial\mathbb{R}^n_+. \end{aligned}\right\}$$

does not possess any positive classical solution.

Theorem 4.4.11. [**51**] *Suppose that $n > 2$. Then the equation*

$$\Delta u + u^p - \mu|\nabla u|^q = 0 \qquad \text{in } \mathbb{R}^n,$$

admits no positive radial ground states if either

$$q > \min\big\{p, 2p/(p+1)\big\},\ 0 < p \leq \frac{n}{n-2} \quad \text{or} \quad q \geq \bar{q},\ \frac{n}{n-2} < p \leq p_S,$$

where $\bar{q}$ is a function of p and n such that $2p/(p+1) < \bar{q} < p$ for $n/(n-2) < p < p_S$ and $\bar{q} = p$ for $p = p_S$.

The proof of the following theorem is completely different from that in Case i, which is based on energy, measure arguments, rescaling and elliptic Liouville-type theorems.

Theorem 4.4.12. *Let $p > 1$, $q \geq 1$, $p > q > 2p/(p+1)$ and $T > 0$. Assume that $p < p_S$ and Ω convex bounded. Then for any nonnegative classical solution of (4.6) on Q_T that satisfies $u_t > 0$, there holds*

$$u(x,t) \leq C(p, \Omega)\big(1 + t^{-q/2(p-q)} + (T-t)^{-q/2(p-q)}\big). \tag{4.49}$$

Proof. Since u is a subsolution of the same problem with $\mu = 0$ and the same initial data (due to $u_t - \Delta u - u^p < u_t - \Delta u - u^p + \mu|\nabla u|^q = 0$). Moreover, $p+1 > n(p-1)/2$ due to $p < p_S$. Then, by Theorem 4.4.9 in view of the comparison principle, it follows that

$$\inf_{t\in(0,\tau/2)} \|u(t)\|_{p+1} \leq C(p,\Omega,\tau) \leq C(p,\Omega)\big(1 + t^{-q/2(p-q)} + (T-t)^{-q/2(p-q)}\big). \quad (4.50)$$

We argue by contradiction and assume that for each $k = 1, 2, ...$ there exists a global solution $u_k \geq 0$ of (4.6) such that

$$\|u_k(t)\|_{p+1} > k \quad \text{ for all } \; t \in (0, \tau/2). \quad (4.51)$$

Denote

$$E_k(t) = E\big(u_k(t)\big) = \frac{1}{2}\int_\Omega |\nabla u_k(t)|^2 \mathrm{d}x - \frac{1}{p+1}\int_\Omega u_k^{p+1}(t)\mathrm{d}x.$$

Recall that $E^{'}(t) = -\int_\Omega u_t^2(t)\mathrm{d}x - \int_\Omega u_t|\nabla u|^q \mathrm{d}x \leq 0$ and that u_k satisfies the identity

$$\begin{aligned}\frac{1}{2}\frac{d}{\mathrm{d}t}\int_\Omega u_k^2 \mathrm{d}x &= \int_\Omega u_k^{p+1}(t)\mathrm{d}x - \int_\Omega |\nabla u_k(t)|^2 \mathrm{d}x - \mu\int_\Omega u_k|\nabla u_k(t)|^q \mathrm{d}x \\ &= -2E_k(t) + \frac{p-1}{p+1}\int_\Omega u_k^{p+1}(t)\mathrm{d}x - \mu\int_\Omega u_k(t)|\nabla u_k(t)|^q \mathrm{d}x.\end{aligned} \quad (4.52)$$

Step 1. We claim that

$$E_k\big(\tau/4\big) \geq k^{1/2} \quad (4.53)$$

for all $k \geq k_0$ large enough.

Assume (4.53) fails. Using (4.52) and Hölder's inequality, we obtain, for all $t > \tau/4$.

$$\frac{1}{2}\frac{d}{\mathrm{d}t}\int_\Omega u_k^2 \mathrm{d}x \geq -2k^{1/2} + \frac{p-1}{p+1}\int_\Omega u_k^{p+1}(t)\mathrm{d}x - \mu\int_\Omega u_k(t)|\nabla u_k(t)|^q \mathrm{d}x,$$

by using (3.12), then we have

$$\frac{1}{2}\frac{d}{\mathrm{d}t}\int_\Omega u_k^2 \mathrm{d}x \geq -2k^{1/2} + \Big(\frac{p-1}{p+1} - \mu C\Big)\int_\Omega u_k^{p+1}(t)\mathrm{d}x, \quad (4.54)$$

hence

$$\frac{1}{2}\frac{d}{\mathrm{d}t}\int_\Omega u_k^2 \mathrm{d}x \geq -2k^{1/2} + C\Big(\int_\Omega u_k^2(t)\mathrm{d}x\Big)^{(p+1)/2}.$$

This implies

$$\int_\Omega u_k^2(t)\mathrm{d}x \le Ck^{1/(p+1)}, \quad t \ge \tau/4, \tag{4.55}$$

since otherwise $\int_\Omega u_k^2(t)\mathrm{d}x$ has to blow up in finite time. Integrating (4.54) over $(\tau/4,\tau/2)$ and using (4.51) and (4.55), we obtain

$$\frac{1}{4}k\tau \le \int_{\tau/4}^{\tau/2}\int_\Omega u_k^{p+1}\mathrm{d}x\mathrm{d}t \le C(k^{1/(p+1)} + k^{1/2}\tau)$$

and this is a contradiction for $k \ge k_0$ large.

Step 2. Let $a > 0$ to be fixed later and set $F_k = \{t \in (0,\tau/4] : \; -E_k^{'} \ge E_k^{1+1/a}(t)\}$. We claim that $|F_k| < \tau/8$ for all $k \ge k_0$ large enough.

We note that $E_k > 0$ on $(0,\tau/4]$ for $k \ge k_0$ by (4.53), since $E_k^{'} \le 0$. By definition of F_k, we have

$$\left(aE_k^{-1/a}\right)^{'} = -E_k^{'}E_k^{-1-1/a} \ge \chi_{F_k} \quad \text{on } (0,\tau/4].$$

By integration, we deduce that $aE_k^{-1/a}\big(\tau/4\big) \ge |F_k|$. Hence by (4.53) we have $|F_k| < \tau/8$.

Step 3. We choose

$$a \ge (p+1)/(p-1). \tag{4.56}$$

We claim that for all $k \ge k_0$ large,

$$\|\partial_t u_k(t)\|_2^2 \le C\Big(\int_\Omega u_k^{p+1}(t)\mathrm{d}x\Big)^{\frac{a+1}{a}} \qquad \text{for all } t \in (0,\tau/4] \setminus F_k. \tag{4.57}$$

For all $t \in (0,\tau/4] \setminus F_k$, we have

$$\|\partial_t u_k(t)\|_2^2 \le \|\partial_t u_k(t)\|_2^2 + \int_\Omega |\nabla u|^q u_t \mathrm{d}x = -E_k^{'}(t) \le E_k^{1+1/a}(t) \le \|\nabla u_k(t)\|_2^{2(1+1/a)}. \tag{4.58}$$

Hence, by (4.52), Hölder's and Young's inequalities, and (3.12)

$$\|\nabla u_k(t)\|_2^2 \le \int_\Omega u_k^{p+1}(t)\mathrm{d}x + \|u_k(t)\|_2\,\|\partial_t u_k(t)\|_2 + \mu\int_\Omega u_k^{p+1}(t)\mathrm{d}x$$

$$\begin{aligned}
&\leq C\int_\Omega u_k^{p+1}(t)\mathrm{d}x + \|u_k(t)\|_2 \|\nabla u_k(t)\|_2^{1+1/a} \\
&\leq C\int_\Omega u_k^{p+1}(t)\mathrm{d}x + C_1 \|u_k(t)\|_{p+1} \|\nabla u_k(t)\|_2^{1+1/a} \\
&\leq C\int_\Omega u_k^{p+1}(t)\mathrm{d}x + C_1 \|u_k(t)\|_{p+1}^{\frac{2a}{a-1}} + \frac{1}{2}\|\nabla u_k(t)\|_2^2 \\
&\leq C_2\int_\Omega u_k^{p+1}(t)\mathrm{d}x + \frac{1}{2}\|\nabla u_k(t)\|_2^2,
\end{aligned}$$

where we used (4.56) and (4.51), hence

$$\|\nabla u_k(t)\|_2^2 \leq C\int_\Omega u_k^{p+1}(t)\mathrm{d}x.$$

This estimate with (4.58) implies (4.57).

Step 4. Let $0 < r < (p+1)/2, \quad b = (p+1-r)(a+1)/a$ and

$$G_k = \{t \in (0, \tau/4] : \quad \|\partial_t u_k(t)\|_2^2 \leq C\|u_k(t)\|_\infty^b\}.$$

We claim that $|G_k| > 0$, where $|G_k|$ means the Lebesgue measure of G_k.

Due to Lemma 4.4.8, for $A = A(p, r, \Omega, \tau) > 0$ large enough, the set

$$\widetilde{G}_k := \{t \in (0, \tau/4] : \int_\Omega u_k^r(t)\mathrm{d}x \geq A\}$$

satisfies

$$|\widetilde{G}_k| < \tau/8, \tag{4.59}$$

We deduce from (4.51) that

$$\int_\Omega u^{p+1}(t)\mathrm{d}x \leq C\|u_k(t)\|_\infty^{p+1-r}\int_\Omega u_k^r(t)\mathrm{d}x \leq C\|u_k(t)\|_\infty^{p+1-r}.$$

Therefore, $G_k \supset (0, \tau/4] \setminus (F_k \cup \widetilde{G}_k)$ by Step 3. The claim then follows from Step 2 and (4.59).

Step 5. Now, we will use rescaling argument to have a contradiction by using Step 4, for each large k, we may pick $t_k \in G_k$. By (4.51), we can choose $x_k \in \overline{\Omega}$ such that $M_k = u_k^{2(p-q)/q}(x_k, t_k)$, denote $\lambda_k = M_k^{-2(p-q)/q}$ and put

$$v_k(y) = \lambda_k^{\frac{q}{2(p-q)}} u_k(x_k + \lambda_k^{1/2} y, t_k),$$

$$\tilde{v_k}(y) = \lambda_k^{\frac{2p-q}{2(p-q)}} \partial_t u_k(x_k + \lambda_k^{1/2} y, t_k).$$

Then the functions $v_k, \widetilde{v}_k$ satisfy

$$\left.\begin{aligned} \lambda_k^{\frac{p(q+1)-2p}{2(p-q)}} \widetilde{v}_k &= \lambda_k^{\frac{p(q+1)-2p}{2(p-q)}} \Delta v_k + v_k^p - \mu |\nabla v_k|^q \qquad \text{in } \Omega_k, \\ v_k &= 0 \qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{on } \Omega_k, \end{aligned}\right\}, \tag{4.60}$$

where $\Omega_k = \frac{\Omega - x_k}{\lambda_k}$. Moreover, $w_k(0) = 1$ and $0 \leq v_k(0) \leq 1 = v_k(0)$. We need to show that the function v_k are locally uniformly Hölder continuous and $\widetilde{v}_k \to 0$ in an appropriate way.

Let $R > 0$, $B_R(x_0) = \{x \in \Omega : \ |x - x_0| < R\}$ and $B_R^k = \{y \in \Omega_k : \ |y| < R\}$. Since $t_k \in G_k$, we have

$$\begin{aligned} \int_{B_R^k} |\widetilde{v}_k|^2 \mathrm{d}y &= \lambda_k^{\frac{2p-q}{p-q}} \int_{B_R^k} |\partial_t u_k(x_k + \lambda_k^{1/2} y, t_k)|^2 \mathrm{d}y \\ &= \lambda_k^{\frac{2p-q}{p-q}} \lambda_k^{-n/2} \int_{B_{R\lambda_k(x_k)}} |\partial_t u_k(x, t_k)|^2 \mathrm{d}x \\ &\leq C M_k^{\frac{-2(2p-q)}{q}} M_k^{\frac{n(p-q)}{q}} M_k^b = C M_k^\gamma \end{aligned}$$

for $k \geq k_0$, where

$$\gamma = -\frac{2(2p-q)}{q} + \frac{a+1}{a}(p+1-r) + \frac{n(p-q)}{q}.$$

Due to $(p-q)/q < (p-1)/2$ and $\frac{2p}{q} - 1 \leq p$ for $2p/(p+1) < q < p$. Furthermore, by taking $\frac{2p}{q} - 1 \leq p$ close to p, r close to $(p+1)/2$, $(p-q)/q$ close to $(p-1)/2$ and a sufficiently large, hence γ will be negative provided $p < (n-1)/(n-3)$. (In particular, it is true due to $p < p_S$ if $n \leq 4$.)

Consequently,

$$\int_{B_R^k} |\widetilde{v}_k(y)|^2 \mathrm{d}y \to 0$$

for any $R > 0$. since $0 \leq v_k \leq 1$ and v_k solves (4.60), standard regularity theory implies that v_k is uniformly bounded in $W^{2,2}(B_R^k)$. Since $W^{2,2}$ is embedded in the space of Hölder continuous functions if $n \leq 3$, we may pass to the limit in (4.60) in order to get a limiting solution $v \geq 0$ satisfying the equation

$$\Delta v + v^p - \mu |\nabla v|^q = 0, \tag{4.61}$$

either in $\mathbb{R}^n$ or in a half-space (and satisfying the homogeneous Dirichlet boundary conditions in the later case). Moreover, $v \leq 1$ and $v(0) = 1$, which contradicts the Liouville-type Theorem 4.4.11 in $\mathbb{R}^n$. Furthermore, v is a supersolution of the same problem with $\mu = 0$ ($0 = \Delta v + v^p - \mu|\nabla v|^q < \Delta v + v^p$), in view of the comparison principle ($\Delta u + u^p > \Delta v + v^p$) and Theorem 4.4.10, we have $v < u = 0$ in $\mathbb{R}_+$, a contradiction □

Remark 4.4.13. *If parabolic Liouville-type Theorem 4.3.6 were known for all* $p < p_S$*, then this would imply Theorem 4.4.12 for all* $p < p_S$ *as well. Conversely, it is clear that the estimate (4.49) implies nonexistence of positive solutions of (4.1). We see that Liouville-type theorem and these universal estimates are thus equivalent. On the other hand, Theorem 4.4.12 guarantees that Theorem 4.3.6 remains true for nontrivial nonnegative classical solutions.*

Chapter 5

The Global Existence of the Positive Solutions

The global existence and blow-up solutions for the parabolic equations with a gradient term have been investigated extensively by many authors. For example, Chipot and Weissler [**11**] studied semilinear parabolic equation subject to the homogeneous Dirichlet boundary condition. Existence of global solutions of parabolic equations with a gradient term depends upon the balance between the power of the gradient term and that of the source nonlinearity. By using comparison principle and constructing self-similar subsolution, they obtained sufficient conditions of global existence and blow-up solutions.

The main purpose of this chapter is to show how the exponents of gradient and nonlinear terms and the geometry of the domain affect the existence of global bounded and unbounded solutions. In Section 5.2 we show that there are bounded global solutions for the semilinear parabolic problem with convective gradient terms for the Cauchy problem, while we prove the global existence of semilinear parabolic problem with dissipative gradient term for small initial data in Section 5.3. Section 5.4 is devoted to study the existence of unbounded global solutions in the domains of infinite inradius.

5.1 Introduction

We will consider in this chapter two problems. The first one is a semilinear parabolic problem with a convective gradient term of Cauchy type, which takes the form

$$\left.\begin{array}{ll} u_t - \Delta u = u^p - a \cdot \nabla(u^q), & x \in \mathbb{R}^n, t > 0 \\ u(x,0) = u_0(x), & x \in \mathbb{R}^n. \end{array}\right\}, \tag{5.1}$$

with $p > 1$, $q \geq 1$ and a is a non zero constant vector in $\mathbb{R}^n$.

The second problem is the semilinear parabolic problem with a dissipative gradient term of Dirichlet type, which is

$$\left.\begin{array}{ll} u_t - \Delta u = u^p - \mu|\nabla u|^q & , x \in \Omega, t > 0 \\ u(x,t) = 0 & , x \in \partial\Omega, t > 0 \\ u(x,0) = u_0(x) & , x \in \Omega. \end{array}\right\}, \tag{5.2}$$

with $p, q > 1$ and $\mu > 0$.

The problem of semilinear convective reaction diffusion equation (5.1) in the one-dimensional case on a bounded interval with homogeneous Dirichlet boundary conditions has been introduced by several authors [**9, 21, 36, 1**]. Their results concerning blow-up and global existence of solutions can be given as follows:

i. if $p > q$ both global and blowing up solution exist, depending on the size of the initial value.

ii. if $p \leq q$ all solutions are global.

Moreover, in [**21**] A. Friedman and A. A. Lacey showed that the problem (5.1) has a single point blow-up for a large class of initial values if $q = 2$ and $p > 3$. Furthermore, the problem (5.1) was considered on a bounded domain of $\mathbb{R}^n$ by A. Friedman in [**20**], he proved that if $p > q > 1$ and the initial value is large enough then the solution blows-up in finite time, which means that, in a bounded domain, if $p > q$ the convective term $a.\nabla(u^q)$ has no effect with respect to the global or local character of the solutions, however, blow-up cannot occur when $p \leq q$. Moreover, it is known in [**55**] that for all nontrivial $u_0 \geq 0$, the blow-up in finite or infinite time can occur for (5.1) whenever $q \geq p > 1$ and $p < p_F$. On the other hand, it was obtained in [**18**] that the blow up rate for the problem (5.1) has also the form

$$u(x,t) \leq C(T-t)^{-\frac{1}{p-1}},$$

in the subcritical case $q < (p+1)/2$.
S. Kaplan has considered in [**29**] the stability of the zero solution $u \equiv 0$ of the problem (5.2) in bounded domains. It has been proved that, the solution is global, bounded and decays exponentially to zero, for all nonnegative data of sufficiently small L^∞ norm.
Moreover, for $\Omega = \mathbb{R}^n$, some kind of stability can be found in [**52**] in the case $q = 2p/(p+1)$, regardless of the sign and the size of μ. It has been shown that the solution of (5.2) is global, decays to zero, and asymptotically self-similar, whenever the initial data are small with respect to a special norm related to the heat semigroup.
On the other hand, it is known in [**60**] that the exact self-similar global solutions are constructed and they take the form

$$u(t,x) = (t+1)^{-1/(p-1)}U(|x|(t+1))^{-1/2}).$$

In the case $q \geq p$, it was shown in [**16, 46**] that for bounded domains, the dissipative term $\mu|\nabla u|^q$ prevents blow-up, neither in finite nor infinite time. Furthermore, the problem (5.2) in [**55**] has been considered in arbitrary unbounded domains when $q \geq p$. It turns out that the geometry of Ω at infinity plays a determining role in the problem, the relevant notion was the inradius of Ω ($\rho(\Omega)$). It was proved that if $\rho(\Omega) < \infty$ then the solution of (5.2) is global and bounded, and if $\rho(\Omega) = \infty$, then there exist (possibly global) unbounded solutions for all $q \geq p$ and $\mu > 0$ (see [**55, 56**]).
Moreover, in [**55**] some results were obtained concerning global existence, boundedness or unboundedness of solutions for the problem (5.2), which are summarized in the following points

- if Ω contains a cone (in particular $\Omega = \mathbb{R}^n$), and $q \geq p$, then there exist unbounded global solutions.
- if $\Omega = \mathbb{R}^n$ and $q \geq p$, then some solutions blow-up in infinite time at every point of $\mathbb{R}^n$, while, if $\Omega \neq \mathbb{R}^n$, blow-up can only occur at infinity.
- in any domain Ω (in particular in $\mathbb{R}^n$), for $q \geq p$, the solution exists globally whenever u_0 has exponential decay in at least one direction.
- If the restriction of u_0 to some cone contained in Ω has a slow enough decay at infinity when $q \geq p$, the the solution blows-up in finite or infinite time.
- If Ω is contained in a strip, the solutions are global and uniformly bounded for all u_0 if $q \geq p$, and for small u_0 if $1 < q < p$ (with μ large if $1 < q < 2p/(p+1)$).
- If $\Omega \neq \mathbb{R}^n$ and $q \geq p$, then the blow-up set of all unbounded solutions (global or not) is $\{\infty\}$. On the other hand, if $\Omega = \mathbb{R}^n$, then the blow-up set for any unbounded solution is either $\mathbb{R}^n \cup \{\infty\}$ or ∞.

The aim of this chapter is to prove that the blow up of the problem (5.1) can occur just in infinite time ($T_{max} = \infty$) for suitably small data. Furthermore, the stability of the zero solution of the problem (5.2) is proved, when the inradius of the domain is infinite. Finally, we show the unboundedness of the solutions of the problem (5.2) in domains of infinite inradius.

5.2 Small Data Global Solutions for the Cauchy Problem

Global existence for the problem (5.1) with data dominated by a small multiple of a Gaussian can be shown by the comparison principle argument, by looking for a supersolution of the form $v(x,t) = t^{\alpha}\tilde{G}(x,t)$, where $\alpha > 0$ and using $\tilde{G}_t - \Delta\tilde{G} = 0$.

Theorem 5.2.1. *Consider the problem (5.1) with $q > p > 1$ and $p < p_F$ then $T_{max}(u_0) = \infty$ for some nontrivial $u_0 \in X_+$.*

Proof. We will build a self-similar supersolution of (5.1) in the form

$$v(x,t) = t^{\alpha}\tilde{G}(x,t)$$

for some $\alpha > \frac{n}{2}$, where $\tilde{G} = (4\pi)^{\frac{n}{2}}G$, and G is the Gaussian heat kernel. By setting $k = \alpha - \frac{n}{2}$, the function v satisfies

$$\begin{aligned} v_t - \Delta v - v^p + a \cdot \nabla(v^q) &= t^{\alpha}(\tilde{G}_t - \Delta\tilde{G}) + \alpha t^{\alpha-1}\tilde{G} - t^{\alpha p}\tilde{G}^p + t^{\alpha q}a \cdot \nabla(\tilde{G}^q) \\ &= \alpha t^{k-1}e^{-\frac{|x|^2}{4t}} - t^{kp}e^{-\frac{p|x|^2}{4t}} - qt^{kq+\frac{1}{2}}(\frac{x \cdot a}{2})e^{-\frac{q|x|^2}{4t}} \\ &\geq (\alpha t^{k-1} - t^{kp} - Ct^{kq+\frac{1}{2}})e^{-\frac{q|x|^2}{4t}}. \end{aligned}$$

Here we used $se^{-\frac{q}{s^2}} \leq Ce^{-s^2}$, $s \geq 0$. Now, since $p < p_F := 1 + \frac{2}{n}$ and ($q > p$, we can choose $q < 1 + \frac{3}{n}$), by taking $\alpha > 0$ sufficiently small, it follows that $kp < k + \frac{2}{n}(\alpha - \frac{n}{2}) = k - 1$ and $kq < k + \frac{3}{n}(\alpha - \frac{n}{2}) = k - \frac{3}{2}$, which is means $kq + \frac{1}{2} < k - 1$, so that $v_t - \Delta v - v^p + a \cdot \nabla(v^q) \geq 0$ in $\mathbb{R}^n$ for $t \geq t_0$, where $t_0 \geq 1$ is large enough [since α is small enough, then $k-1$ is negative]. If $u_0(x) \leq t_0^{-\frac{n}{2}}e^{-\frac{|x|^2}{4t_0}}$, the comparison in proposition D.1.5 then guarantees that $u(t) \leq v(t_0+t)$ on $[0, T_{max}(u_0))$ and u exists globally. $\square$

5.3 Small Data Global Solutions for the Dirichlet Problem

In Theorem 5.3.3 below, we shall need the notion **asymptotically stable solution** which is defined as follows:

Definition 5.3.1. *Assume that $f(0) = 0$ (so that $u \equiv 0$ is a solution to (4.8)) and that (4.8) is locally well-posed in a space X. We say that the zero solution is **asymptotically stable** in X if there exists a constant $\eta > 0$ such that, for all $u_0 \in X$ with $\|u_0\|_X \leq \eta$, there holds $T_{max} = \infty$ and*

$$\lim_{t\to\infty} \|u(t)\|_X = 0.$$

In order to prove that the zero solution is asymptotically stable in L^1 if Ω has infinite inradius, we need to recall the following proposition which has been proved by Souplet and Weissler in (p. 349, [**55**]).

Proposition 5.3.2. *Suppose the regular domain Ω contains a cone, $\mu \geq 0$, $q \geq p > 1$. There exists some $u_0 \in C^2(\bar{\Omega})$, $u_0 \geq 0$, with compact support, such that the solution u of (5.2) satisfies $T_{max} = \infty$ and*

$$\lim_{t\to\infty} \|u(t)\|_\infty = \infty.$$

Theorem 5.3.3. *Consider the problem (5.2) with $q \geq p > 1$ and $\mu > 0$.*
Assume that $p < 1 + p_F$. If $\rho(\Omega) = \infty$, then there is exist initial data $u_0 \in L^1(\Omega)$ of arbitrary small L^1-norm such that $T_{max}(u_0) = \infty$.

Proof. Fix a test function $\phi \in D(\mathbb{R}^n)$, $\phi \geq 0$, $\phi \neq 0$ with $supp(\phi) \subset B := B(0, 1)$, and let w be the solution of (5.2) with Ω replaced by B and u_0 replaced by ϕ.
Due to Proposition 5.3.2, we can assume that w blows-up in infinite time.
Now since $\rho(\Omega) = \infty$, Ω contains some ball $B_k(x_k, k)$ for any integer $k \geq 1$.
Let us set

$$u_k(x,t) = k^{\frac{-2}{p-1}} w\Big(\frac{x - x_k}{k}, \frac{t}{k^2}\Big), \qquad u_{0,k} = k^{\frac{-2}{p-1}} \phi\Big(\frac{x - x_k}{k}\Big).$$

Due to the invariance of the equation under this scaling, it is easily verified that u_k solves the problem

$$\left.\begin{aligned} \partial_t u_k - \Delta u_k &= u_k^p - \mu k^{\frac{q(p+1)-2p}{p-1}} |\nabla u_k|^q && , x \in B_k,\ t > 0 \\ u_k &= 0 && , x \in \partial B_k,\ t > 0 \\ u_k(x,0) &= u_{0,k}(x) && , x \in B_k. \end{aligned}\right\} \tag{5.3}$$

Let $\tilde{u}_k$ be the solution of the problem (5.2) with $u_0 = u_{0,k}$. Since B_k is included in Ω and $\tilde{u}_k \geq 0$ on ∂B_k, it follows that the equation (5.3) for $k \to \infty$ become

$$\left.\begin{aligned} \partial_t u_k - \Delta u_k - u_k^p + \mu |\nabla u_k|^q = -\mu(k^{\frac{q(p+1)-2p}{p-1}} - 1)|\nabla u_k|^q = -\infty < 0,&& x \in B_k,\ t > 0 \\ u_k = 0,&& x \in \partial B_k,\ t > 0 \\ u_k(x,0) = u_{0,k}(x) = 0,&& x \in B_k. \end{aligned}\right\}$$

Then, by the comparison principle that $\tilde{u}_k \geq u_k$, hence $\tilde{u}_k$ blows-up in infinite time. Last, an easy calculation yields

$$\|u_{0,k}\|_1 = \|k^{\frac{-2}{p-1}}\|_1 \|\phi\|_1 = k^{\frac{-2}{p-1}+n} \|\phi\|_1 \to 0 \quad \text{as} \quad k \to \infty.$$

$\square$

5.4 Global Unbounded Solutions in a Domain with Infinite Inradius

The proof of unbounded global existence of Theorem 5.4.1 is based on a comparison argument, by constructing a stationary supersolution v in the exterior of ball of small radius ε, which is radial and whose minimum is larger than $\|u_0\|_\infty$. The solution u is dominated by v, centered at points y such that $B(y,\varepsilon) \subset \Omega^c$. Since $\rho(\Omega) = \infty$, then any point x of Ω is at unbounded distance of such a point y. This guarantees a unboundedness for u.

Theorem 5.4.1. *Consider the problem (5.2) with $q \geq p > 1$, $\mu > o$. Assume $\rho(\Omega) = \infty$. Then there exists $u_0 \in X_+$, such that $T_{max}(u_0) = \infty$ and $\lim\limits_{t\to\infty} \|u(t)\|_\infty = \infty$.*

Proof. We choose $\varepsilon \in (0,1)$ such that for a ball of radius $\tilde{\rho}(\Omega)$, $B \cap \Omega^c$ contains a ball of radius ε, $\tilde{\rho}(\Omega) \geq \rho(\Omega)$.
Let a be a fixed point in Ω and we choose x_a such that

$$B(x_a, \varepsilon) \subset \Omega^c$$

and

$$|x_a - a| \leq \tilde{\rho}(\Omega). \tag{5.4}$$

We seek for a supersolution of (5.2) of the form $v(x,t) = Ke^{\alpha r}$, $r = |x - x_a|$, $\alpha \geq 0$. The inequality $Pv := v_t - \Delta v + \mu|\nabla v|^q - v^q \geq 0$ needs to be checked if $r \geq \varepsilon$. To ensure that, it must

$$0 - \alpha^2 K e^{\alpha r} - \alpha \frac{n-1}{r} K e^{\alpha r} + \mu \alpha^q K^q e^{\alpha q r} - K^p e^{\alpha p r} \geq 0, \quad r > \varepsilon,$$

which is satisfied if

$$\mu \alpha^q K^{q-1} e^{\alpha(q-1)r} \geq K^{p-1} e^{\alpha(p-1)r} + \alpha^2 + \alpha \frac{n-1}{\varepsilon}, \quad r > \varepsilon.$$

Since $q \geq p > 1$, then we have

$$\mu\alpha^q K^{q-1} \geq 2K^{p-1} \quad \text{and} \quad \mu\alpha^q K^{q-1} \geq 2\alpha^2 + 2\alpha\frac{n-1}{\varepsilon}.$$

It thus suffices to choose $\alpha = (2/\mu)^{1/q}$ and next

$$K = \max\{\|u_0\|_\infty, 1, (\alpha^2 + \alpha(n-1)/\varepsilon)^{1/(q-1)}\}.$$

Then, it follows from comparison principle that $0 \leq u(x,t) \leq v(x,t)$ in Ω, as long as $u(t)$ exists. In particular, by using (5.4), we obtain

$$0 \leq u(a,t) \leq K \exp[\tilde{\rho}(\Omega) = \infty] = \infty.$$

Since a was an arbitrary point in Ω, we deduce that $u(t)$ remains unbounded in L^∞ on its existence interval. This implies unbounded global existence. $\square$

Chapter 6

Viscous Hamilton-Jacobi Equations (VHJ)

VHJ is the simplest type of a parabolic PDE with a nonlinear term depending on the first order spatial derivative of u, where in [**30**],[**32**] VHJ is presented in the physical theory of growth and roughening of the surfaces, which is defined as Kardar-Parisi-Zhang equation. The blow-up phenomenon of this problem is different from the equations with a nonlinearity depending on u, where the function u itself remains uniformly bounded, but its gradient blows-up in finite time, and this phenomenon is called gradient blow-up (GBU).

The main purpose of this chapter is to study whether the speed of divergence of GBU of Dirichlet problem for VHJ with $p > 2$, specially the upper GBU rate estimate in n space dimension is the same as in one space dimension. In section 6.2 we consider the upper estimates of the blow-up profile of ∇u for the solutions of (6.1). Next we shall consider in section 6.3 the upper GBU rate estimate for the problem (6.2) in a convex bounded domain.

6.1 Introduction

Consider the following initial-boundary value problems with zero Dirichlet boundary condition

$$\left.\begin{aligned} u_t - \Delta u = |\nabla u|^p, &\qquad x \in \Omega, \quad t > 0, \\ u = 0, &\qquad x \in \partial\Omega, \quad t > 0, \\ u(x,0) = u_0, &\qquad x \in \Omega. \end{aligned}\right\} \tag{6.1}$$

and

$$\left.\begin{aligned} u_t - \Delta u = |\nabla u|^p + \lambda, &\qquad x \in \Omega, \quad t > 0, \\ u = 0, &\qquad x \in \partial\Omega, \quad t > 0, \\ u(x,0) = u_0, &\qquad x \in \Omega, \end{aligned}\right\} \tag{6.2}$$

where Ω is a bounded convex domain, $u_0 \in X_+$, $p > 2$ and $\lambda > 0$.

The main idea of these problems is that the function u itself remain uniformly bounded, but its gradient goes to infinity in finite time T. In other words,

$$\|\nabla u(t)\|_\infty \to \infty, \qquad \text{whereas} \qquad \sup_{t\in[0,T)} \|u(t)\|_\infty < \infty.$$

The gradient blow-up phenomena has been studied for the first time in [**19**] by Filippov, he considered a one dimensional problem with time-depending Dirichlet boundary condition.

The problem (6.1) when $p \leq 2$ was introduced in [**47, 59**], and they proved that all solutions are global, i.e., $T = \infty$. However, if $p > 2$, then the GBU in finite time is known to occur (see [**59**], [**57**]). For instance, in [**57**] Souplet showed that the singularity may come from the suitably large initial data in finite time. Moreover, it has been shown in [**47**] in one space dimension and $\Omega = (0,1)$, that the bounds of blow-up profile of u_x take the form

$$u_x(x,t) \leq U'(x) + C_1 x, \qquad 0 < x \leq 1 \tag{6.3}$$

$$\text{and} \qquad u_x(x,t) \geq -U'(1-x) - C_1(1-x), \qquad 0 \leq x < 1$$

The function $U \in C([0,\infty)) \cap C^1((0,\infty))$ is a solution (a singular steady state) of

$$V'' + V'^p = 0, \quad x > 0, \qquad V(0) = 0.$$

$$\text{and} \qquad U := d_p x^{(p-2)/(p-1)}, \qquad U^{'}(x) := d_p^{'} := x^{-1/(p-1)}, \quad x > 0,$$

where $d_p = (p-2)^{-1}(p-1)^{(p-2)/(p-1)}$ and $d_p^{'} = (p-1)^{-1/(p-1)}$, which shows that the bounds on blow-up profile of u_x is away from $x = 0$ and 1, and this means that GBU may occur just on the boundary.

On the other hand, the lower GBU rate estimates for this problem have been considered by Guo and Hu in [**28**], it was shown that the lower GBU rate estimate take the form

$$\sup_{s\in[0,t]} \|\nabla u(t)\|_\infty \geq C(T-t)^{-1/(p-2)}, \qquad t \to T.$$

Furthermore, the analogue of the lower estimates is also true in one space dimension (see [**14**]), namely

$$\|u_x(t)\|_\infty \geq C(T-t)^{-1/(p-2)}, \qquad t \to T.$$

Moreover, under additional assumptions $u_{0,xx} + |u_{0,xx}|^p \geq 0$ and $u_0 \in C^2([0,1])$, the corresponding upper GBU rate estimate can be established (see [**28**]) as follows

$$u_x(0,t) \leq C(T-t)^{-1/(p-2)}, \qquad t \to T. \tag{6.4}$$

Similar result was obtained in [**47**] for the closely related one-dimensional problem

$$\left.\begin{aligned} u_t - u_{xx} &= |u_x|^p + \lambda, & & x \in (0,1), \quad t > 0, \\ u &= 0, & & x \in \{0,1\}, \quad t > 0, \\ u(x,0) &= u_0, & & x \in (0,1). \end{aligned}\right\}$$

with $p > 2$, $\lambda > 0$ and $u_0 \in X_+$.

The aim of this section is to show that the results of Quittner and Souplet in [**47**] of the upper estimates of the blow-up profile (6.3) and the time rate of gradient blow up (6.4) hold true for problem (6.2) in n-dimension.

6.2 Upper Profile Estimates of GBU

This subsection considers the profile estimate of the gradient solutions of problem (6.1), which shows that the gradient blow-up cannot occur in the interior of the domain, as will be shown in Theorem 6.2.3.

In order to prove that, we need to recall the following lemma, which shows some properties of the time-derivative u_t, which has been proved in [**47**],[**59**].

Lemma 6.2.1. *Consider problem (6.1) with* $p > 1$ *and* $u_0 \in X_+$*, and let* $0 < t_0 < T := T_{max}(u_0)$*. There exists* $C_1 > 0$ *such that*

$$|u_t| \leq C_1, \qquad x \in \Omega, \quad t_0 \leq t < T. \tag{6.5}$$

Proof. The function $w := u_t$ satisfies

$$\left.\begin{aligned} w_t - \Delta w &= a(x,t) \cdot \nabla w, \qquad && x \in \Omega, \quad 0 < t < T, \\ w &= 0, && x \in \partial\Omega, \quad 0 < t < T, \end{aligned}\right\} \tag{6.6}$$

where

$$a(x,t) = p|\nabla u|^{p-2}\nabla u. \tag{6.7}$$

By parabolic regularity results, we have $u_t \in C^{2,1}(Q_T)$, and due to (1.19) we have $u_t \in BC(\bar{\Omega} \times [t_0, t_1]), \ \ 0 < t_0 < t_1 < T$. As an consequence of (6.6) and of the maximum principle in Proposition D.1.1 we obtain the upper estimate of u_t. □

The proof of Theorem 6.2.3 relies on using a modification of the Bernstein technique and a suitable cut-off function, with considering a partial differential equation satisfied by $|\nabla u|^2$. We follow the procedure which has been used in [**59**].

Let $x_0 \in \Omega$ be fixed, $0 < t_0 < T < T_{max}(u_0)$, $R > 0$ such that $B(x_0, R) \subset \Omega$ and we write $Q_{T,R}^{t_0} = B(x_0, R) \times (t_0, T)$.

Let $\alpha \in (0,1)$ and set $R' = \frac{3R}{4}$. We select a cut-off function $\eta \in C^2(\bar{B}(x_0, R'))$, $0 < \eta < 1$, with $\eta(x_0) = 1$ and $\eta = 0$ for $|x - x_0| = R'$, such that

$$\left.\begin{aligned} &|\nabla \eta| \leq CR^{-1}\eta^{\alpha} \\ &|D^2\eta| + \eta^{-1}|\nabla\eta|^2 \leq CR^{-2}\eta^{\alpha} \end{aligned}\right\} \qquad \text{for } |x - x_0| < R'$$

where $C = C(\alpha) > 0$.

In order to show that the GBU in problem (6.1) occurs only on the boundary, we need to state the following lemma.

Lemma 6.2.2. *Let u_0, u be positive solutions of (6.1). We denote $w = |\nabla u|^2$ and $z = \eta w$. Then at any point $(x_1, t_1) \in Q^{t_0}_{T,R'}$, such that $|\nabla u(x_1, t_1)| > 0$, z is smooth and satisfies the following differential inequality*

$$\mathcal{L}z + Cz^p \leq C_1 + CR^{\frac{-2p}{p-1}},$$

$$\mathcal{L}z = \partial_t z - \Delta z - H \cdot \nabla z,$$

H is defined by (6.9).

Proof. We know that $\nabla u \in C^{2,1}$ in a neighbourhood of such points and hence we can differentiate the equation (6.1).

Assume that $w = |\nabla u|^2$ satisfies the differential equation

$$\partial_t w - \Delta w - (pw^{\frac{p-2}{2}} \nabla u) \cdot \nabla w = -2|D^2 u|^2.$$

Indeed, for $i = 1, ..., N$, put $u_i = \frac{\partial u}{\partial x_i}$ and $w_i = \frac{\partial w}{\partial x_i}$. Differentiating (6.1) in x_i, we have

$$\partial_t u_i - \Delta u_i = \frac{p}{2} w^{\frac{p-2}{2}} w_i. \tag{6.8}$$

Multiplying (6.8) by $2u_i$, summing up, and using $\Delta w = 2\nabla u \cdot \nabla(\Delta u) + 2|D^2 u|^2$, we deduce that

$$\mathcal{L}w = -2|D^2 u|^2,$$

where

$$H = pw^{\frac{p-2}{2}} \cdot \nabla u. \tag{6.9}$$

Setting $z = \eta w$, we get

$$\mathcal{L}z = \mathcal{L}(\eta w) = \partial_t(\eta w) - \Delta(\eta w) - H \cdot \nabla(\eta w)$$

$$= \eta \mathcal{L} w + w \mathcal{L} \eta - 2 \nabla \eta \cdot \nabla w.$$

Now we shall estimate the different terms. In what follows $\delta > 0$ can be chosen arbitrary small.

- Estimate $|2\nabla\eta \cdot \nabla w|$.

  Using Young's inequality, we have

  $$|2\nabla\eta \cdot \nabla w| \leq C\eta^{-1}|\nabla\eta|^2 w + \delta\eta|D^2 u|^2,$$

  where we used the fact $\nabla w = 2D^2 u \nabla u$.

- Estimate $|wH \cdot \nabla\eta|$.

  $$\begin{aligned} |wH \cdot \nabla\eta| &= |wpw^{\frac{p-2}{2}} \nabla u \cdot \nabla\eta| \\ &= |pw^{\frac{p}{2}} w^{\frac{1}{2}} \nabla\eta| \\ &= pw^{\frac{p+1}{2}} |\nabla\eta|. \end{aligned}$$

Finally choosing $\delta = 1$, we have

$$\mathcal{L} z + \eta |D^2 u|^2 \leq C(p,n) w[|\Delta\eta| + \eta^{-1}|\nabla\eta|^2] + C|\nabla\eta| w^{\frac{p+1}{2}}.$$

Using the properties of the cut-off function η, we get

$$\mathcal{L} z + \eta |D^2 u|^2 \leq C(p,n) w R^{-2} \eta^{\alpha} + C R^{-1} \eta^{\alpha} w^{\frac{p+1}{2}}. \tag{6.10}$$

Using the result of Lemma 6.2.1, we shall estimate $|D^2 u|^2$ in terms of a power of w. For $(x_1, t_1) \in Q^{t_0}_{T,R'}$, such that $|\nabla u(x_1, t_1)| > 0$, we have

$$\begin{aligned} |\nabla u(x_1, t_1)|^p &= \partial_t u(x_1, t_1) - \Delta u(x_1, t_1) \\ &\leq C_1 + \sqrt{N} |D^2 u(x_1, t_1)|. \end{aligned}$$

Hence

$$\frac{1}{N} |\nabla u(x_1, t_1)|^{2p} \leq C_1 + |D^2 u(x_1, t_1)|^2.$$

There are two cases

$$\text{either} \qquad \frac{1}{N}|\nabla u(x_1,t_1)|^{2p} \le 2C_1,$$

$$\text{or} \qquad \frac{1}{N}|\nabla u(x_1,t_1)|^{2p} \le |D^2u(x_1,t_1)|^2.$$

In both cases we arrive at

$$\frac{1}{C(N,p)}|\nabla u(x_1,t_1)|^{2p} \le C_1 + |D^2u(x_1,t_1)|^2.$$

Using this inequality, it follows from (6.10) that, at (x_1,t_1)

$$\mathcal{L}z + \frac{\eta}{C(N,p)}|\nabla u(x_1,t_1)|^{2p} - C_1\eta \le \mathcal{L}z + \eta|D^2u|^2$$

$$\le C(N,p)wR^{-2}\eta^{\alpha} + CR^{-1}\eta^{\alpha}w^{\frac{p+1}{2}}$$

Hence

$$\mathcal{L}z + \frac{\eta}{C(N,p)}w^p \le C_1 + CwR^{-2}\eta^{\alpha} + CR^{-1}\eta^{\alpha}w^{\frac{p+1}{2}}.$$

We take $\alpha = \frac{p+1}{2p} \in (0,1)$. Using Young's inequality, we have

$$\begin{aligned} CR^{-1}\eta^{\frac{p+1}{2p}}w^{\frac{p+1}{2}} &\le CR^{\frac{-2p}{p-1}} + \frac{1}{4C}\eta w^p, \\ CR^{-2}\eta^{\frac{p+1}{2p}}w &\le CR^{\frac{-2p}{p-1}} + \frac{1}{4C}\eta^{\frac{p+1}{2}}w^p. \end{aligned}$$

Using that $\eta \le 1$, we get

$$\mathcal{L}z + \frac{\eta}{C(N,p)}w^p \le C_1 + CR^{\frac{-2p}{p-1}} + \frac{1}{2C}\eta|\nabla u|^{2p}.$$

Hence

$$\mathcal{L}z + \frac{1}{2C(N,p)}z^p \le C_1 + CR^{\frac{-2p}{p-1}}. \tag{6.11}$$

□

Theorem 6.2.3. *Let $p > 2$, $M > 0$ and $u_0 \geq 0$, $\|\nabla u_0\|_{L^\infty} \leq M$. Let u be solution of (6.1), then*

$$|\nabla u| \leq C_2 \delta^{-\frac{1}{p-1}}(x) + C_3 \quad \text{in } \ \Omega \times (0, T_{max}(u_0)).$$

This means the blow up may only take place on the boundary.

Proof. First let us note that by the local existence, there exists $t_0 \in (0, T_{max}(u_0))$ with $t_0 = t_0(M, p, N)$ such that

$$\sup_{0 \leq t \leq t_0} \|\nabla u\|_{L^\infty} \leq C(p, \Omega, M). \tag{6.12}$$

We also know that ∇u is a locally Hölder continuous function and thus z is a continuous function on $\overline{B(x_0, R')} \times [t_0, T] = \overline{Q}$, for any $T < T_{max}(u_0)$. Therefore, z must reach a positive maximum at some point $(x_1, t_1) \in \overline{B(x_0, R')} \times [t_0, T]$, unless $z = 0$ in $\overline{Q}$. Since $z = 0$ on $\partial B_{R'} \times [t_0, T]$, we deduce that $x_1 \in B_{R'}$. Therefore, $\nabla z(x_1, t_1) = 0$ and $D^2 z(x_1, t_1) \leq 0$. Now we have either $t_1 = t_0$ or $t_0 < t_1 < T$.
If $t_0 = t_1$, then

$$z(x_1, t_1) = |\nabla u(x_1, t_1)| \, |\eta| \leq \|\nabla u(x_1, t_0)\|^2_{L^\infty}.$$

If $t_0 < t < T$, we have $\partial_t z(x_1, t_1) \geq 0$ and therefore $\mathcal{L}z \geq 0$. Using (6.11) we obtain,

$$\frac{1}{2C(p, N)} z^p(x_1, t_1) \leq C_1 + C R^{\frac{-2p}{p-1}},$$

which is

$$\sqrt{z(x_1, t_1)} \leq C_1 + C R^{\frac{-1}{p-1}}.$$

Since $z(x_0, t) \leq z(x_1, t_1)$ and $\eta(x_0) = 1$ we get,

$$|\nabla u(x_0, t)| \leq C_1 + C R^{\frac{-1}{p-1}} \qquad \text{for } t \in [t_0, T].$$

By taking $R = \delta(x_0)$, letting $T \to T_{max}(u_0)$ and using (6.12) we have,

$$|\nabla u| \leq C_2 \delta^{\frac{-1}{p-1}}(x_0) + C_3.$$

□

6.3 Blow-up Rate Estimate

Theorem 6.3.3 below considers the upper bound of the blow-up rate for problem (6.2), following the procedure used in [**28**], which used a suitable auxiliary function with the application of the maximum principle.

In order to prove Theorem 6.3.3, we need to recall the following results which have been proved in [**47**].

Proposition 6.3.1. *Assume $p > 1$ and $u_0 \in X_+$. Let u be the solution of (6.1) and let $0 < T < T_{max}$. Then*

$$\sup_{t\in[0,T]} \|\nabla u(t)\|_\infty = \sup_{\mathcal{P}_T} |\nabla u|.$$

Theorem 6.3.2. *Consider problem (6.1) with $p > 2$ and $\Omega \neq \mathbb{R}^n$. Let $u_0 \in X_+$ and assume that $T := T_{max}(u_0) < \infty$. Then there exists $C > 0$ such that*

$$\sup_{x\in[0,t]} \|\nabla u(s)\|_\infty \geq C(T-t)^{-1/(p-2)}, \qquad t \to T. \tag{6.13}$$

Proof. Denote

$$m(t) := \sup_{\overline{\Omega}\times[T/2,t]} |\nabla u| = \max_{s\in[T/2,t]} \|\nabla u(s)\|_\infty\,, \qquad T/2 \leq t < T.$$

Step 1. We claim that $w := u_t$ satisfies

$$\|\nabla w(t)\|_\infty \leq Cm^{p-1}(t), \qquad T/2 < t < T. \tag{6.14}$$

Let $t \in (T/2, T)$, $s \in (T/4, t)$, and put $K = \sup_{\sigma\in[0,t-s]} \sigma^{1/2} \|\nabla w(s+\sigma)\|_\infty$. For $\tau \in (0, t-s)$, in view of (6.6), (6.7) and variation-of-constants formula, we have

$$w(s+\tau) = e^{-\tau A}w(s) + \int_0^\tau e^{-(\tau-\sigma)A}(a\cdot\nabla w)(s+\sigma)\mathrm{d}\sigma.$$

Using Proposition C.2.3, Lemma 6.2.1, and the fact that $\int_0^\tau (\tau-\sigma)^{-1/2}\sigma^{-1/2}\mathrm{d}\sigma = \int_0^1 (1-z)^{-1/2}z^{-1/2}\mathrm{d}z$, it follows that

$$\|\nabla w(s+\tau)\|_\infty \leq C\tau^{-1/2} \|w(s)\|_\infty + C\int_0^\tau (\tau-\sigma)^{-1/2} \|a\cdot\nabla w(s+\sigma)\|_\infty \mathrm{d}\sigma$$

$$\leq C\tau^{-1/2} + Cm^{p-1}(t)K.$$

Multiplying by $\tau^{1/2}$ and taking the supremum for $\tau \in [0, t-s]$, we obtain

$$K \leq C + C(t-s)^{1/2} m^{p-1}(t)K.$$

Now choosing $s = t - (1/4)\min(T, (Cm^{p-1}(t))^{-2}) \in (T/4, t)$, we obtain $K \leq 2C$, hence

$$\|\nabla w(t)\|_\infty \leq 2C(t-s)^{-1/2} \leq 4C \max(T^{-1/2}, Cm^{p-1}(t)).$$

Since m is positive nondecreasing, this implies Claim (6.14).

Step 2. We next claim that m is locally Lipschitz on $(T/2, T)$ and that

$$m'(t) \leq Cm^{p-1}(t), \quad \text{for a.e. } t \in (T/2, T). \tag{6.15}$$

Let $T/2 < t < s < T$. For any $\tau \in [t, s]$ and $x \in \Omega$, it follows that the mean-value inequality and (6.14) that

$$|\nabla u(x,\tau) - \nabla u(x,t)| \leq (\tau - t) \sup_{\Omega \times [t,\tau]} |\partial_t \nabla u| \leq C(\tau - t)m^{p-1}(\tau),$$

hence

$$|\nabla u(x,\tau)| \leq |\nabla u(x,t)| + C(\tau - t)m^{p-1}(\tau) \leq m(t) + C(s-t)m^{p-1}(s).$$

Taking the supremum for (x, τ) over $\Omega \times [t, s]$, we get

$$0 \leq m(s) - m(t) \leq C(s-t)m^{p-1}s.$$

Since m is continuous, the claim follows.

Finally, integrating (6.15) over (t, s) with $T/2 < t < s < T$, and using $m(s) \to \infty$ as $s \to T$, we infer that

$$m(t) \geq C(T-t)^{-1/(p-2)},$$

which implies estimate (6.13). $\square$

Theorem 6.3.3. *Consider the problem (6.2) with $p > 2$ and $\lambda > 0$. Let Ω be a convex bounded domain and let $u_0 \in X_+ \cap C^2(\overline{\Omega})$ be such that*

$$\Delta u_0 + |\nabla u_0|^p + \lambda \geq 0 \qquad \text{in } \overline{\Omega}.$$

If $T := T_{max}(u_0) < \infty$, then there exists $C > 0$ such that

$$\|\nabla u(t)\|_\infty \leq C(T-t)^{-\frac{1}{p-2}}, \qquad t \to T.$$

Proof. For any $\eta > 0$ is small enough, set

$$\Omega^\eta := \{x \in \Omega : \ d(x, \partial\Omega) > \eta\}.$$

we shall derive a lower bounded on u_t away from the parabolic interior of $\Omega \times (0, T)$. We consider the parabolic operator

$$\mathcal{L}\phi := \phi_t - \Delta\phi - p|\nabla u|^{p-2}\nabla u \nabla\phi.$$

For $\sigma \in (0, 1)$ and $\eta \in (0, T)$ to be chosen later, we introduce the function

$$w(x,t) = \Big(1 + \frac{1}{m^\sigma(t)}\Big)\Big(1 - \frac{|\nabla u|}{m(t)}\Big), \qquad x \in \Omega^\eta \times (\eta, T),$$

where

$$m(t) = \max_{x \in \Omega^\eta \times [\eta, t]} |\nabla u(x,t)|, \qquad \text{as } \ t \to T. \tag{6.16}$$

Step1. We shall show that for suitable $\eta \in (0, T)$ and $C > 0$, there holds

$$w + u \leq C u_t \qquad \text{in } \ \Omega^\eta \times (\eta, T). \tag{6.17}$$

We may assume $m(t) \geq 1$ without loss of generality. By the proof of Theorem 6.3.2, m is locally Lipschitz on $(T/2, T)$ and (6.15) is satisfied.

A straight computation gives:

$$\mathcal{L}w = \Big(1 + \frac{1}{m^\sigma}\Big)\frac{|\nabla u| m'}{m^2} - \frac{\sigma m'}{m^{\sigma+1}}\Big(1 - \frac{|\nabla u|}{m}\Big). \tag{6.18}$$

Since $m' \geq 0$ a.e., we have

$$\mathcal{L}w = \frac{m'}{m^{\sigma+1}}\Big(-\sigma + (\sigma+1)\frac{|\nabla u|}{m} + \frac{|\nabla u|}{m^{1-\sigma}}\Big) \leq \frac{m'}{m^{\sigma+1}}\Big(-\sigma + (\sigma+2)\frac{|\nabla u|}{m^{1-\sigma}}\Big) < 0,$$

in case $|\nabla u(x,t)| < \frac{\sigma}{\sigma+2} m^{1-\sigma(t)}$.

However, if $|\nabla u(x,t)| \geq \frac{\sigma}{\sigma+2} m^{1-\sigma(t)}$, then by (6.18) and (6.15), we have

$$\mathcal{L}w \leq \Big(1 + \frac{1}{m^\sigma}\Big)\frac{|\nabla u| m'}{m^2} \leq C|\nabla u|\frac{m^{p-1}}{m^2} \leq \frac{C}{m}\Big(\frac{\sigma+2}{\sigma}\Big)^{\frac{p-2}{1-\sigma}} |\nabla u|^{\frac{p-1-\sigma}{1-\sigma}}.$$

If we choose $\sigma = 1/(p-1)$, then $(p-1-\sigma)/(1-\sigma) = p$. Then

$$\mathcal{L}(w+u) \leq \frac{\tilde{C}}{m}|\nabla u|^p - (p-1)|\nabla u|^p + \lambda, \qquad \tilde{C} := C\Big(\frac{\sigma+2}{\sigma}\Big)^{\frac{p-2}{1-\sigma}}.$$

Hence, for t close to T and by (6.16) we obtain

$$\mathcal{L}(w+u) \leq -\frac{(p-1)}{2}|\nabla u|^p + \lambda, \qquad a.e. \text{ in } \ \Omega^\eta \times (\eta, T).$$

If $|\nabla u(x,t)|^p \geq 2\lambda/(p-1)$, then $\mathcal{L}(w+u) \leq 0$.
In both cases we obtain $\mathcal{L}(w+u) \leq 2\lambda(w+u)$, hence:

$$\mathcal{L}(e^{-2\lambda t}(w+u)) \leq 0 = \mathcal{L}u_t \qquad \text{in } \ \Omega^\eta \times (\eta, T). \tag{6.19}$$

Due to Theorem 6.2.3, assume that $x_0 \in \partial\Omega$ is a GBU point. By Proposition 6.3.1 that

$$m(t) = \|\nabla u(t)\|_\infty = |\nabla u(x_0, t)|, \qquad \eta < t < T, \tag{6.20}$$

by taking η closer to T if necessary. Thus,

$$[w+u](x,t) = 0 = u_t(x,t), \quad x \in \partial\Omega^\eta, \tag{6.21}$$

and by applying maximum principle to u_t we also have $u_t(x,\eta) > C > 0$ on the parabolic boundary of $\Omega^\eta \times (\eta, T)$, and by Theorem 6.2.3, if $\eta > 0$ is small enough then there exists $C_0 > 0$ such that $w \leq -C_0 < 0$ on $\{x : \ x = \eta\}$, and since $u \leq C_0$ on $\{x : \ x = \eta\}$, then there exists $C > 0$ such that

$$[e^{-2\lambda t}(w+u) - Cu_t](\eta, t) \leq 0 \qquad x \in \partial\Omega^\eta. \tag{6.22}$$

Moreover, there exists $C > 0$ such that

$$[e^{-2\lambda\eta}(w+u) - Cu_t](\eta, t) \leq 0 \qquad x \in \Omega^\eta. \tag{6.23}$$

Using (6.19), (6.21), (6.22) and (6.23) and maximum principle in Proposition D.1.6, we deduce $e^{-2\lambda t}(w+u) \leq Cu_t$ in $\Omega^\eta \times (\eta, T)$, hence $u + w \leq Cu_t$.

Step2. As a result from (6.17) and (6.20), we have

$$\begin{aligned} C|\nabla u_t(x_0,t)| &= \lim_{h\to 0} \frac{C|u_t(x_0+h,t)|}{|h|} \geq \lim_{h\to 0} \frac{\big|[w+u](x_0+h,t)\big|}{|h|} \\ &\geq \lim_{h\to 0} \frac{|w(x_0+h,t)|}{|h|} = \nabla|w(x_0,t)| \geq \Big(1+\frac{1}{m^\sigma(t)}\Big)\Big(\frac{|\Delta u(x_0,t)|}{m(t)}\Big) \\ &\geq \frac{|\nabla u(x_0,t)|^p}{m(t)} = |\nabla u(x_0,t)|^{p-1}. \end{aligned}$$

By integration from t to T we obtain

$$|\nabla u(x_0,t)| \leq C(T-t)^{-\frac{1}{p-2}}, \qquad t \to T.$$

□

Chapter 7

Conclusions

This thesis is devoted to investigate the possible effect of the gradient term depending nonlinearities, on the global existence or the nonexistence and the asymptotic behaviour of the solutions of semilinear elliptic and parabolic equations, with the presence (or not) of the reaction term u^p. For the elliptic problems whose nonlinearity depends on u and on the spatial derivative of u, we consider questions about the nonexistence of the solutions under certain conditions on the exponents of nonlinearities in the whole space. Concerning parabolic problems with gradient terms, it is known that the solution may cease to exist in a finite time: The solution blows up, where we considered the nature of blow-up set and the rate estimate of blow-up solutions. However, we showed that there are problems of this type that have global solutions. Moreover, for the parabolic problems, we considered the nonexistence of nontrivial bounded solutions which are defined for all negative and positive times on the whole space (Liouville-type theorems). The gradient blow-up behaviour was also studied for the viscous Hamilton-Jacobi problem with Dirichlet boundary condition in n-dimension space.
The conclusion for every chapter can be summarized as follows:

1. For the quasilinear elliptic equation (2.1) defined in $\mathbb{R}^n$, it has been proven in [**51**] that the conditions $q > 2p/(p+1)$ and $p \leq n/(n-2)$, $n > 2$ imply the nonexistence of the positive radial ground states. This result can be extended

to show that the nonexistence also holds when $q > 2p/(p+1)$ and $p \le p_F$.

2. For the Dirichlet problem (3.3) defined in a convex domain, we showed that if $q > 2p/(p+1)$, then the set of blow-up points is a compact subset of the domain. Furthermore, we established the blow-up rate estimates for this problem when $q > 2p/(p+1)$ in a ball and in a convex domain, showing that the upper blow-up rate estimates are more singular than those known for the problem (3.3) when $q < 2p/(p+1)$ in both domains.

3. For the parabolic equation with a dissipative gradient term (4.1), which is defined in the whole space $\mathbb{R}^{n+1} = \mathbb{R}^n \times \mathbb{R}$, we proved the Liouville-type theorems under the condition $q > 2p/(p+1)$, in radial case when $p < p_F$, and in general case when $p < p_B$. Moreover, we showed that the universal bounds for the global solutions and the usual blow up rate estimates for the problem (4.5) when $q = 2p/(p+1)$, is the same as that known when $q < 2p/(p+1)$. On the other hand, we proved that the universal bounds for the problem (4.5) when $q < 2p/(p+1)$ become false for stronger perturbation terms, i.e., when $q > 2p/(p+1)$.

4. For the Cauchy problem for the semilinear parabolic problem with a convective gradient term (5.1), with a suitable positive small data, we proved that the problem admits a bounded global solution when $q > p > 1$ and $p < p_F$. Moreover, for Dirichlet problem for the semilinear parabolic problem with a dissipative gradient term defined in infinite inradius domain, and small initial data, we showed that the zero solution is asymptotically stable and the blow-up occurs in infinite time, if $q \ge p > 1$ and $p < p_F$. However, we showed that if $p < p_F$, then the problem (5.2) which is defined in infinite inradius domain, has unbounded global solutions for some positive initial data.

5. For the viscous Hamilton-Jacobi problem (6.1) defined in a convex domain subset of $\mathbb{R}^n$, with the zero Dirichlet boundary condition, and nonnegative initial data, it has been shown in [**47**] that, the gradient blow-up points of

the problem (6.1) in one space dimension may occur only on the boundary. We extended this result to the higher dimension, proving that the gradient blow-up cannot occur in the interior of the domain. Moreover, we showed that the upper estimate of the gradient blow-up in one space dimension is still true in the higher dimension.

Appendix A

Notation

A.1 Geometric notation

Let Ω be a domain, nonempty, connected, open subset of $\mathbb{R}^n$ and let $k \in \mathbb{N}$, then

(i) We write $\Omega' \subset\subset \Omega$ if the closure of Ω' is a compact subset of Ω.

(ii) Ω is a uniformly regular of class C^k, if either $\Omega = \mathbb{R}^n$ or there exists a countable family (U_j, φ_j), $j = 1, 2, ...$ of coordinate charts with the following properties:

  (a) Each φ_j is a C^k-diffeomorphism of U_j onto the open unit ball B_1 in $\mathbb{R}^n$ mapping $U_j \cap \Omega$ onto the upper half-ball $B_1 \cap (\mathbb{R}^{n-1} \times (0, \infty))$ and $U_j \cap \partial\Omega$ onto the flat part $B_1 \cap (\mathbb{R}^{n-1} \times \{0\})$. In addition, the function φ_j and the derivatives of φ_j and φ_j^{-1} up to the order k are uniformly bounded on U_j and B_1, respectively.

  (b) The set $\bigcup_j \varphi_j^{-1}(B_{1/2})$ contains an ε-neighbourhood of $\partial\Omega$ in $\overline{\Omega}$ for some $\varepsilon > 0$.

(iii) $\delta(x) := \mathrm{dist}(x, \partial\Omega)$.

(iv) Inradius $\rho(\Omega)$ of a domain Ω is defined by

$$\rho(\Omega) = \sup\{r > 0 : \Omega \ \text{ contains a ball of radius } \ r\} = \sup_{x \in \Omega} \mathrm{dist}(x, \partial\Omega).$$

Moreover, strict inradius $\tilde{\rho}(\Omega) \geq \rho(\Omega)$ is defined by

$$\tilde{\rho}(\Omega) = \inf\{R > 0 : \exists\varepsilon > 0 \text{ such that for any ball } B \text{ of radius } R,$$
$$B \cap \Omega^c \text{ contains a ball of radius } \varepsilon\}.$$

(v) $\nu(x)$ is the exterior unit normal on $\partial\Omega$ at a point $x \in \partial\Omega$

(vi) $Q_T := \Omega \times (0, T)$, for $0 < T < \infty$,
$S_T := \partial\Omega \times (0, T)$, for $0 < T < \infty$,
$\mathcal{P}_T := S_T \cup (\overline{\Omega} \times \{0\})$, for $0 < T < \infty$.

A.2 Notation for functions

(i) By a solution of a PDE problem being positive, we mean that $u(x) > 0$ or $u(x, t) > 0$ in the domain under consideration.

(ii) We say that a domain is symmetric if either $\Omega = \mathbb{R}^n$, or $\Omega = B = \{x \in \mathbb{R}^n |\, |x| < R\}$, or $\Omega = B_{R,R'} = \{x \in \mathbb{R}^n |\, R < |x| < R'\}$, where $0 < R < R' \leq \infty$. Where B_R is an open ball in $\mathbb{R}^n$ with centre zero and radius R.

(iii) Denote $r = |x|$ and $J \subset \mathbb{R}$ be an interval. A function u defined on a symmetric domain Ω (resp., on $\Omega \times J$) is said to be radially symmetric. or simply radial, if it can be written in the form $u = u(r)$ (resp., $u = u(r, t)$ for each J).

(iv) The function u is said to be radial nonincreasing, if it is radial, and if u is nonincreasing as a function or r.

(v) The equilibrium solution of a PDE problem means that $u_t = 0$.

A.3 Function space notation

i. $C(\Omega) = \{u : \Omega \to \mathbb{R} |\, u \text{ continuous}\}$
$C(\overline{\Omega}) = \{u \in C(\Omega) |\, u \text{ is uniformly continuous on bounded subsets of } \Omega\}$.

$C^k(\Omega) = \{u : \Omega \to \mathbb{R} |\ u$ is k-times continuously differentiable$\}$.

$C^\infty(\Omega) = \{u : \Omega \to \mathbb{R} |\ u$ is infinitely differentiable$\}$.

The support of a function u, denoted by Supp(u) is the closure of the set $\{x : u(x) \neq 0\}$.

$\mathcal{D}(\Omega)$, denote the functions in C^∞ with compact support.

ii. $L^p(\Omega) = \{u : \Omega \to \mathbb{R} |\ u$ is Lebesgue measurable, $\|u\|_{L^p(\Omega)} < \infty\}$, where

$$\|u\|_{L^p(\Omega)} = \Big(\int_\Omega |f|^p \mathrm{d}x \Big)^{1/p}, \quad 1 \leq p < \infty.$$

iii. $L^p_{loc}(\Omega) = \{u :\to \mathbb{R} |\ v \in L^p(\Omega')$ for each $\Omega' \subset\subset \Omega\}$.

iv. Let $1 \leq p \leq \infty$ and k is a nonnegative integer, then the Sobolev space $W^{k,p}(\Omega)$ is the space of functions $u \in L^p(\Omega)$ satisfying $D^\alpha u \in L^p(\Omega)$, $|\alpha| \leq k$.

v. $W^{k,p}_{loc}(\Omega) = \{u \in L^p(\Omega) |\ D^\alpha u \in L^p_{loc}(\Omega)$ for all $|\alpha| \leq k\}$.

vi. $W^{1,2}_0(\Omega)$ is the closure of $\mathcal{D}(\Omega)$ in $W^{1,2}(\Omega)$. In other words, $W^{1,2}_0(\Omega)$ is comprising the functions $u \in W^{1,2}(\Omega)$ such that $u = 0$ on $\partial\Omega$. $W^{1,2}_0(\Omega)$ will be denoted as Hilbert space $H^1_0(\Omega)$.

vii. We denoted by $BC^{2,1}(Q)$ the space of functions $u \in BC(Q)$ whose second derivative in Ω and first derivative in $(0, T)$ are bounded and continuous. t.

viii. A function $u \in C(\Omega)$ is said to be Hölder continuous of order $\alpha \in (0, 1)$ if

$$H_\alpha \equiv \sup\{|u(x) - u(\xi)|/|x - \xi|^\alpha |\ x, \xi \in \Omega \text{ and } x \neq \xi\} < \infty.$$

The Hölder norm of u is defined by

$$|u|_\alpha = \|u\|_\infty + H_\alpha = \sup_{x \in \Omega} |u(x)| + H_\alpha.$$

The set of all Hölder-continuous functions in Ω with finite norm is denoted by $BUC^{\alpha}(\Omega)$. Note that if Ω is compact, then $BUC^{\alpha}(\Omega) = C^{\alpha}(\Omega)$.

When the domain Ω is replaced by Q_T we define the Hölder constant by

$$H_{\alpha} \equiv \sup\{|u(x,t) - u(\xi,s)|/(|t-s|^{\alpha/2} + |x-\xi|^{\alpha})|\ (t,x),(\xi,s) \in Q_T\}.$$

The Hölder norm of u is given by

$$|u|_{\alpha} = \|u\|_{\infty} + H_{\alpha} = \sup_{(x,t)\in Q_T} |u(x,t)| + H_{\alpha}.$$

The set of all Hölder-continuous functions in Q_T with finite Hölder norm is denoted by $C^{\alpha,\alpha/2}(Q_T)$.

Let k be a nonnegative integer, $\alpha \in (0,1)$ and $a = k+\alpha$. Then we define the set of functions in $C^{\alpha}(Q_T)$ with finite norms

$$|u|_{1+\alpha} \equiv \sup_{Q_T} |u| + \sum |D_x u|_{\alpha} + |u_t|_{\alpha}$$

$$|u|_{2+\alpha} \equiv \sup_{Q_T} |u| + \sum |D_x u|_{\alpha} + \sum |D_x^2 u|_{\alpha} + |u_t|_{\alpha}$$

$$\vdots$$

$$|u|_a = |u|_{k+\alpha} \equiv \sup_{Q_T} |u| + \sum |D_x u|_{\alpha} + \sum |D_x^2 u|_{\alpha} + ... + \sum |D_x^k u|_{\alpha} + |u_t|_{\alpha}$$

are denoted by $C^{a,a/2}(Q_T)$ or $BUC^{a,a/2}(Q_T)$.

ix. $\mathcal{D}(\Omega \times (0,T))$ is the space of C^{∞}-functions with compact support in $\Omega \times (0,T)$.

x. If $Q \subseteq \mathbb{R}^n \times \mathbb{R}$ is a domain in space and time, then

$$C^{2,1}(Q) = \{u : Q \to \mathbb{R}|\ u, D_x u, D_x^2 u, u_t \in C(Q)\}.$$

xi. $L^{\infty}(Q_T) = \{u : Q_T \to \mathbb{R}|\ u$ is Lebesgue measurable, $\|u\|_{L^{\infty}(Q_T)} < \infty\}$, where

$$\|u\|_{L^{\infty}(Q_T)} = ess \sup_{Q_T} |u|.$$

xii. $L^p(Q_T) = \{u : Q_T \to \mathbb{R} |\ u$ is Lebesgue measurable, $\|u\|_{L^p(Q_T)} < \infty\}$,
where

$$\|u\|_{L^p(Q_T)} = \Big(\iint |u|^p \mathrm{d}x\mathrm{d}t\Big)^{1/p} \qquad 1 \leq p < \infty.$$

xiii. $L^p_{loc}(Q_T) = \{u : Q_T \to \mathbb{R} |\ v \in L^p(Q'_T)$ for each $Q'_T \subset\subset Q_T\}$.

xiv. If $Q \subseteq \mathbb{R}^n \times \mathbb{R}$, we denote by the Sobolev space $W^{2,1;p}(Q)$, the space of functions $u \in L^p(Q)$ satisfying $u_t, D_x u, D_x^2 \in L^p(Q)$, endowed with the norm

$$\|u\|_{2,1;p} = \|u\|_{2,1;p;Q} := \|u\|_{p;Q} + \|D_x u\|_{p;Q} + \left\|D_x^2 u\right\|_{p;Q} + \|u_t\|_{p;Q}\,.$$

xv. $W^{2,1;p}_{loc}(Q_T) = \{u \in L^p(Q_T) |\ u_t, D_x u, D_x^2 u \in L^p_{loc}(Q_T)\}$.

Appendix B

Basic inequalities

In this appendix we will recall some basic inequalities as:

Young's inequality

Let Ω be an arbitrary domain in $\mathbb{R}^n$, $1 < p < \infty$, $\varepsilon > 0$ and $q = p/(p-1)$. Then

$$xy \leq \frac{\varepsilon^p x^p}{p} + \frac{\varepsilon^{-q} y^q}{q} \qquad x, y > 0.$$

Hölder's inequality

Let $1 \leq p \leq \infty$ and $q = p/(p-1)$. Then

$$\|uv\|_1 \leq \|u\|_p \|v\|_q, \qquad u \in L^p(\Omega), \quad v \in L^q(\Omega).$$

Jensen's inequality

Assume that $F : \mathbb{R} \to [0, \infty)$ is a convex function, and that $w : \Omega \to [0, \infty]$ is measurable and satisfies $\int_\Omega w(x)\mathrm{d}x = 1$. If u is a measurable function on Ω such that $uw, F(u)w \in L^1(\Omega)$, then

$$F\Big(\int_\Omega u(x)w(x)\mathrm{d}x \Big) \leq \int_\Omega F(u(x))w(x)\mathrm{d}x.$$

Appendix C

Fundamental materials for elliptic and parabolic equations

In this appendix we list some essential estimates and some notations of solutions of parabolic equations.

C.1 Model elliptic problems

C.1.1 Elliptic regularity

Consider the problem

$$Au = f \qquad \text{in } \Omega, \tag{C.1}$$

where $f = f(x)$ is a given function, A is the second-order elliptic differential operators of the form

$$Au = -\sum_{i,j=1}^{n} a_{ij} \frac{\partial^2}{\partial x_i \partial x_j} u + \sum_{i=1}^{n} b_i \frac{\partial}{\partial x_i} u + cu, \tag{C.2}$$

with measurable coefficients a_{ij}, b_i, c satisfying the condition

$$\sum_{i,j} a_{ij}(x)\xi_i\xi_j \geq \lambda|\xi|^2, \qquad \text{for all } x \in \Omega,\ \xi \in \mathbb{R}^n, \lambda > 0, \tag{C.3}$$

with a uniform bound

$$|a_{ij}|, |b_i|, |c| \leq \Lambda, \quad \Lambda > 0. \tag{C.4}$$

C.1.2 Strong solutions

The strong solution of (C.1) is a function $u \in W^{2,1}_{loc}$ which satisfies the equation (C.1) a.e.

C.1.3 Interior-boundary elliptic L^p-estimates

Theorem C.1.1. *Let Ω be an arbitrary domain in $\mathbb{R}^n$ and assume (C.3) and (C.4) are satisfied. Let $u \in W^{2,p}_{loc} \cap L^p(\Omega)$, $1 < p < \infty$, be a strong solution of (C.1), where a_{ij} are continuous and $f \in L^p(\Omega)$. Let Σ be an open subset of $\partial\Omega$ of class C^2, $u \in W^{2,p}(\Omega)$ and $u = 0$ on Σ. Let $a_{ij} \in C(\Omega \cup \Sigma)$ and $\Omega' \subset\subset \Omega \cup \Sigma$. Then*

$$\|u\|_{2,p;\Omega'} \leq C(\|u\|_p + \|f\|_p), \tag{C.5}$$

where C depends on $n, p, \Omega, \Omega', \lambda, \Lambda$, the continuity of the a_{ij} on Ω, and on Σ.

C.2 Model parabolic problems

C.2.1 Parabolic regularity notations

Consider the problem

$$u_t + Au = f \qquad \text{in } Q_T, \tag{C.6}$$

Where $f = f(x)$ is a given function, A is defined in (C.2), and the coefficients depend on $(x,t) \in Q_t$,

$$\sum_{i,j} a_{ij}(x,t)\xi_i\xi_j \geq \lambda|\xi|^2, \qquad \text{for all } (x,t) \in Q_T,\ \xi \in \mathbb{R}^n, \lambda > 0, \tag{C.7}$$

C.2.2 Strong solutions

The strong solution of (C.6) is a function $u \in W^{2,1;1}_{loc}(Q_T)$ satisfying (C.6) a.e.

C.2.3 Interior-boundary parabolic L^p-estimates

Theorem C.2.1. *Let Ω be an arbitrary domain in $\mathbb{R}^n$ and assume (C.7) and (C.4) are satisfied. Let $u \in W^{2,1;p}_{loc} \cap L^p(Q_T)$, $1 < p < \infty$, be a strong solution of (C.6), where $a_{ij} \in C(\overline{Q_T})$ and $f \in L^p(Q_T)$. Let Ω be of class C^2 and either Σ be an open subset of S_T or $\Sigma = \mathcal{P}_T$. Assume $u \in W^{2,1;p}(Q_T)$ and $u = 0$ on Σ. Let $Q' \subset Q_T$,* $\operatorname{dist}(Q', \mathcal{P}_T \setminus \Sigma) > 0$ *if $\Sigma \neq \mathcal{P}_T$. Then*

$$\|u\|_{2,1;p;Q'} \leq C(\|u\|_{p;Q_t} + \|f\|_{p;Q_t}), \tag{C.8}$$

where C depends on $n, p, Q_t, Q', \lambda, \Lambda$, the continuity of the a_{ij}, and on Σ.

C.2.4 The variation-of-constants formula

Definition C.2.2. *The variation of constants formula of the solution of the problem (C.6) is defined as*

$$u(t) = e^{-(t-\tau)A}u(\tau) + \int_\tau^t e^{-(t-s)A} f(u(s))\mathrm{d}s, \qquad 0 < \tau < t < T,$$

e^{-tA} is the Dirichlet heat semigroup in Ω.

C.2.5 Gradient estimate for the heat semigroup

The following proposition from [**34**] gives the smooth estimate for the gradient term of the heat semigroup.

Proposition C.2.3. *Let Ω be a domain of class $C^{2+\alpha}$ for some $\alpha \in (0,1)$ and let $\left(e^{-tA}\right)_{t\geq 0}$ be the Dirichlet heat semigroup in Ω. For all $\Phi \in L^\infty(\Omega)$, there holds*

$$\left\|\nabla e^{-tA}\Phi\right\|_\infty \leq C(\Omega)(1 + t^{-1/2}) \left\|\Phi\right\|_\infty, \qquad t > 0.$$

C.2.6 Embedding theorem

The following embedding theorem is from [**34**] and [**39**].

Theorem C.2.4. *If $p > n + 2$, $a < 2 - (n + 2)/p$ and Ω is smooth enough, then*

$$W^{2,1;p}(Q) \to BUC^{\alpha,\alpha/2}(Q), \qquad \alpha \in (0, 1). \tag{C.9}$$

C.2.7 Notation for solutions of parabolic problems

C.2.7.1 Classical solutions

Let X be a given space of functions defined in Ω, $u_0 \in X$ and $T \in (0, \infty]$, we say that the function $u \in C([0,T), X)$ is a solution or a classical X-solution of a parabolic problem in $[0,T)$ if $u \in C^{2,1}(\Omega \times (0,T)) \cap C(\overline{\Omega} \times (0,T))$, $u(0) = u_0$ and u is a classical solution of the problem for $t \in (0,T)$.

C.2.7.2 Well-Posedness of the parabolic problem

We say that the parabolic problem is well-posed in X if, given $u_0 \in X$, there exist $T > 0$ and a unique classical X-solution of the problem in $[0,T]$.

C.2.7.3 Maximal solutions

Definition C.2.5. *Let X be a given space of functions defined in Ω. Assume that we have a parabolic problem possesses for each $u_0 \in X$ a unique classical solution u in $[0,T]$, where $T = T(u_0)$. Then there exists $T_{max} = T_{max}(u_0) \in (T, \infty]$ with the following properties.*

(i) The solution u can be continued (in a unique way) to a classical solution on the interval $[0, T_{max})$.

(ii) If $T_{max} < \infty$, then u cannot be continued to a classical solution on $[0,\tau)$ for any $\tau > T_{max}$.

(iii) Assume that $T = T(\|u_0\|_X)$. Then

$$\textit{either} \quad T_{max} = \infty \quad \textit{or} \quad \lim_{t \to T_{max}} \|u(t)\|_X = \infty.$$

C.2.7.4 Weak solutions

Definition C.2.6. *We may define a weak solution of the parabolic problem*

$$\left.\begin{aligned} u_t - \Delta u = F(u, \nabla u), \qquad & x \in \Omega, t > 0, \\ u(x,t) = 0, \qquad & x \in \partial\Omega, t > 0, \\ u(x,0) = u_0(x), \qquad & x \in \Omega. \end{aligned}\right\}, \tag{C.10}$$

on $[0,T]$ *to be a function* $u \in C([0,T]; L^1(\Omega))$ *such that* $f(u) \in L^1(Q_T)$ *and such that the equality*

$$\int_\Omega u(x,t_2)\phi(x,T_2)\mathrm{d}x - \int_\Omega u(x,t_1)\phi(x,T_1)\mathrm{d}x - \int_{t_1}^{t_2}\int_\Omega u\phi_t \mathrm{d}x\mathrm{d}t = \int_{t_1}^{t_2}\int_\Omega (u\Delta\phi + f\phi)\mathrm{d}x\mathrm{d}t,$$

holds for every $\phi \in C^2(\overline{Q}_T)$ *with* $\phi = 0$ *in* $\partial\Omega \times [0,T]$ *and* $0 \le t_1 \le t_2 \le T$.

C.2.7.5 Supersolutions and subsolutions

Definition C.2.7. *A function* $\tilde{u} \in C([0,T] \times \overline{\Omega}) \cap C^{1,2}((0,T] \times \Omega)$ *is called a supersolution or (subsolution) of the problem*

$$\left.\begin{aligned} u_t - \Delta u = F(u, \nabla u) \qquad & in \;\; (0,T] \times \Omega \\ u = 0 \qquad & on \;\; (0,T] \times \; \partial\Omega \\ u(0,x) = u_0(x) \qquad & in \;\; \Omega, \end{aligned}\right\}$$

if it satisfies the inequalities

$$\left.\begin{aligned} \tilde{u}_t - \Delta\tilde{u} \ge \;\; (\le) \;\; F(\tilde{u}, \nabla\tilde{u}) \qquad & in \;\; (0,T] \times \Omega \\ \tilde{u} \ge \;\; (\le) \;\; 0 \qquad & on \;\; (0,T] \times \; \partial\Omega \\ \tilde{u}(0,x) \ge \;\; (\le) \;\; u_0(x) \qquad & in \;\; \Omega. \end{aligned}\right\}$$

Appendix D

Maximum and comparison principles and zero number

D.1 Maximum and comparison principles

In this section we will present maximum and comparison principles, which we frequently used in this study.
The following proposition from [**47**] is a basic maximum principle for classical solutions:

Proposition D.1.1. *Let Ω be an arbitrary domain in $\mathbb{R}^n$, $T > 0$, $b : Q_T \to \mathbb{R}^n$, $c : Q_T \to \mathbb{R}$, with $\sup_{Q_T} c < \infty$. Assume that $w = w(x,t) \in C^{2,1}(Q_T) \cap C(\overline{Q_T})$ satisfies $w \leq 0$ on $\mathcal{P}_T$, $\sup_{Q_T} w < \infty$, and*

$$w_t - \Delta w \leq b \cdot \nabla w + cw \qquad \text{in } Q_T.$$

If Ω is unbounded, assume in addition that either

$$\lim_{|x| \to \infty} \sup_{(x,t) \in Q_T} w(x,t) \leq 0,$$

or

$$|b(x,t)| \leq C_1(1 + |x-a|^{-1}), \qquad x \in Q_T,$$

for some $a \in \mathbb{R}^n$ and $C_1 > 0$. Then $w \leq 0$ in Q_T.

The following proposition (see [**47**]) is a version of the comparison principle for classical (sup-/super-) solutions

Proposition D.1.2. *Let Ω be an arbitrary domain in $\mathbb{R}^n$, $T > 0$, $u, v \in C^{2,1}(Q_T) \cap C(\overline{\Omega}_T)$. Assume that $u \le v$ on $\mathcal{P}_T$ and*

$$\partial_t u - \Delta u - f(x, u, \nabla u) \le \partial_t v - \Delta v - f(x, v, \nabla v) \qquad \text{in } Q_T, \tag{D.1}$$

where $f = f(x, s, \xi) : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ is continuous in x and C^1 in s and ξ. Assume also that

$$u, v, \nabla v \in L^\infty(Q_T), \qquad |u|, |v| \le C_1, |\nabla v| \le C_2$$

and

$$|f_s(x, s, \xi)| + (1 + |x|)^{-1} |f_\xi(x, s, \xi)| \le C_f \qquad \text{for all } |s| \le C_1, |\xi| \le C_2 + 1.$$

Then $u \le v$ in Q_T.

Remark D.1.3. *In proposition (D.1.2) it is sufficient to assume that (D.1) holds in $\tilde{Q}_T := \{(x, t) \in Q_T : u(x, t) > v(x, t)\}$.*

The following proposition from [**47**] is a version of the strong Hopf comparison principle for general semilinear parabolic equations.

Proposition D.1.4. *Let Ω be a bounded domain in $\mathbb{R}^n$ of class C^2, $p > n + 2$, and $T > 0$. Let $u, v \in W^{2,1;p}_{loc}(\overline{\Omega} \times (0, T]) \cap C([0, T], L^2(\Omega)) \cap L^\infty(Q_T)$. Assume*

$$\partial_t u - \Delta u - f(x, t, u, \nabla u) \le \partial_t v - \Delta v - f(x, t, v, \nabla v) \qquad \text{in } Q_T,$$

where $f = f(x, t, s, \xi) : \overline{\Omega} \times [0, T] \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ is continuous in x, t and C^1 in s and ξ. Assume also that $u(., 0) \le v(., 0)$, $u(., 0) \not\equiv v(., 0)$, and either

$$u \le v \qquad \text{on } S_T$$

or

$$\partial_\nu u + bu \le \partial_\nu v + bv \qquad \text{on } S_T, \tag{D.2}$$

where $b \in C^1(\partial\Omega)$. *Finally, if* f *depends on* ξ, *we also assume that* $\nabla u \nabla v \in L^\infty(Q_T)$. *Then*

$$u < v \qquad \text{in } Q_T.$$

In addition, if $u(x_0,t_0) = v(x_0,t_0)$ *for some* $x_0 \in \partial\Omega$ *and* $t_0 \in (0,T)$, *then*

$$\partial_\nu u(x_0,t_0) > \partial_\nu v(x_0,t_0).$$

If (D.2) is true, then $u < v$ *in* $\overline{\Omega} \times (0,T)$.

Next proposition from [**47**] is a weak version of the comparison principle.

Proposition D.1.5. *Let* Ω *be a bounded domain of class* C^2 *or* $\Omega = \mathbb{R}^n$. *Let* $T > 0$ *and* $f, g : (t,u) \in [0,T] \times \mathbb{R} \to \mathbb{R}$ *be such that* f, f_u, g, g_u *are continuous. Let* $u, v \in C^{2,1}(\overline{\Omega} \times (0,T)) \cap L^\infty(Q_T)$. *If* $\Omega = \mathbb{R}^n$ *assume in addition that* $\nabla u \nabla v \in L^\infty_{loc}((0,T), L^\infty(\mathbb{R}^n))$. *If* $u \le v$ *on* S_T, $\limsup_{t\to 0}(u-v)(x,t) \le 0$ *for all* $x \in \Omega$, *and*

$$\partial_t u - \Delta u - f(t,u) - div(g(t,u)) \le \partial_t v - \Delta v - f(t,v) - div(g(t,v)) \qquad \text{in } Q_T,$$

then $u \le v$ *in* Q_T.

The next two propositions from [**47**] are a version of the weak maximum principle, which apply to $W^{2,1;2}_{loc}$ sub-/supersolutions.

Proposition D.1.6. *Let* $0 < T < \infty$. *Let* Ω *be an arbitrary domain in* $\mathbb{R}^n$, c *be measurable and a.e. finite on* Q_T *with* $\sup_{Q_T} C < \infty$, *and* $K \ge 0$. *Assume that* $w \in C(\overline{\Omega} \times (0,T)) \cap C([0,T), L^2_{loc}(\overline{\Omega}))$ *satisfies*

$$\sup_{Q_T} w < \infty, \qquad w_t, \nabla w, D^2 w \in L^2_{loc}(Q_T).$$

If $w \le 0$ *on* $\mathcal{P}_T$ *and*

$$w_t - \Delta w \le K|\nabla w| + cw \qquad \text{a.e. in } Q_T,$$

then

$$w \le 0 \qquad \text{in } Q_T.$$

Proposition D.1.7. *Let $0 < T < \infty$, Ω be an arbitrary domain in $\mathbb{R}^n$, and let $f = f(s,\xi) : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$, be a C^1-function. Let $u \in C(\overline{\Omega} \times (0,T))$ satisfy*

$$u \in C([0,T], L^2_{loc}(\overline{\Omega})), \qquad u \in L^\infty(Q_T), \qquad u_t, \nabla u, D^2 \in L^2_{loc}(Q_T),$$

and similarly for v. If f depends on ξ, we also assume that $\nabla u, \nabla v \in L^\infty(Q_T)$. If $u \le v$ on $\mathcal{P}_T$ and

$$u_t - \Delta u - f(u, \nabla u) \le v_t - \Delta v - f(v, \nabla v) \qquad \text{a.e. in } Q_T,$$

then

$$u \le v \qquad \text{in } Q_T.$$

D.2 Zero number

In this section we define the zero number of functions, and we present the zero number argument which is restricted to one-dimensional or radially symmetric problems.

Definition D.2.1. *The zero number of a function $\psi \in C((0,R))$ is defined as the number of sign changes of ψ in $(0,R)$;*

$$z(\psi) = z_{[0,R]}(\psi) = \sup\{k \in \mathbb{N} : \text{there are } 0 < x_0 < x_1 < ... < x_k < R \\ \text{such that } \psi(x_i)\psi(x_{i+1}) < 0 \text{ for } 0 \le i < k\}.$$

Let $B_R = \{x \in \mathbb{R}^n : |x| < R\}$, $t_1 < t_2$, $q \in L^\infty(B_R, (t_1,t_2))$, $u \in C(\overline{B}_R \times [t_1,t_2]) \cap W^{2,1;\infty}(B_R \times (t_1,t_2))$ and

$$u_t - \Delta u = qu \qquad \text{a.e.in } B_R \times (t_1,t_2). \tag{D.3}$$

Assume that $q(.,t)$ and $u(.,t)$ are radially symmetric for all t, hence $q(x,t) = Q(|x|,t)$ and $u(x,t) = U(|x|,t)$. Then

$$U_r - U_{rr} - \frac{n-1}{r} U_r = QU, \qquad r \in (0,R),\ t \in (t_1,t_2),$$

and $U_r(0,t) = 0$ for all $t \in (t_1,t_2)$.

Theorem D.2.2. *Let q, u be as above, $u \not\equiv 0$, and either $U(R,t) = 0$ for all $t \in [t_1, t_2]$ or $U(R,t) \neq 0$ for all $t \in [t_1, t_2]$. Let $z = z_[0, R]$ denote the zero number in $(0, R)$. Then*

(i) $z\big(U(.,t)\big) < \infty$ *for all* $t \in (t_1, t_2)$,

(ii) *the function* $t \longmapsto z\big(U(.,t)\big)$ *is nonincreasing,*

(iii) *if* $U(r_0, t_0) = U_r(r_0, t_0)$ *for some* $r_0 \in [0, R]$ *and* $t_0 \in (t_1, t_2)$, *then* $z\big(U(.,t)\big) > z\big(U(.,s)\big)$ *for all* $t_1 < t < t_0 < s < t_2$.

Remark D.2.3. *The assertion of Theorem D.2.2 remains true for more general problems of the form*

$$u_t - \Delta u = qu + bx \cdot \nabla u,$$

where $b \in W^{1,\infty}(B_R \times (t_1, t_2))$, $b(x,t) = B(|x|, t)$. *This follows from the fact that the function* $v(x,t) := e^{\frac{1}{2}\int_0^{|x|} B(\xi,t)\xi \mathrm{d}\xi} u(x,t)$ *solves a problem of the form (D.3).*

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